Lattice Theoretic and Logical Aspects

of

Elementary Topoi

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Introduction

Elementary topoi [5], [15], [10], [4], [16], [26] are categories with many aspects in common with the category of sets, but they also generalize sheaf categories $Top(X)$ as well as categories of continuous discrete representations of topological groups $B_{\mathfrak{G}}$, and many others [1], [15], [9], [16], [26].

Any elementary topos $\underline{\mathbf{E}}$ has the rudiments of algebra, i.e. all finite inverse limits, and in the same way as the classical logic is coded into the category of sets by means of the two element set 2 we have an intuitionistic logic built into the subobject classifier Ω in $\underline{\mathbf{E}}$, but the way mathematics develop in $\underline{\mathbf{E}}$ is in the interaction of these concepts and higher order which lives in $\underline{\mathbf{E}}$ in the form of power objects, much like power sets in $\underline{Sets}$.

For various purposes one may assume that elementary topoi have additional properties such as satisfying the axiom of infinity, being Boolean or even satisfying the axiom of choice [15], but here we shall primarily be interested in the properties indicated above and which are common to all elementary topoi.

Any construction which can be performed in an elementary topos $\underline{\mathbf{E}}$ in accordance with the axioms generally allows an internal form, i.e. a combinator which is a morphism in $\underline{\mathbf{E}}$ constructed out of the axioms in such a way that evaluating the combinator on global sections yields cases of the original construction and such that the combinator is preserved by logical functors, in particular by the functors $(\ ) \times A : \underline{\mathbf{E}} \to \underline{\mathbf{E}}/A$. It follows from the principle of extensionality for categories that these properties characterize the combinators. Furthermore, the essential properties of a construction are coded into its combinator in an equational or equally well understood way.

We shall apply the combinators extensively in the presentation. The idea is that we may as well construct the combinators and derive their properties directly. The constructions in question can whenever they are needed always be obtained be evaluation. E. g. to see that $\underline{\mathbf{E}}$ has epi-mono-factorization we construct the combinator associated with this property, namely the internal existential quantification which to a morphism $f : A \to B$ assigns the internal functor $\exists_f : P(A) \to P(B)$ given explicitly as the composite

$$\exists_f = P(A) \xrightarrow{\uparrow seg_{P(A)}} PP(A) \xrightarrow{PP(f)} PP(B) \xrightarrow{\bigcap_B} P(B)$$

and characterized by $\exists_f \dashv P(f)$, i.e. the internal existential quantification along f is left adjoint to the internal substitution along f, and we may define the image of f by the equation:

$$\ulcorner ch_B(im(f)) \urcorner = \ulcorner true_A \urcorner \circ \exists_f .$$

The virtue of this approach is that we are now able to single out the essential points in the arguments via the properties of the combinators, and most proofs reduce to equational calculations, to comparison of inequalities or to applying the uniqueness theorem for adjoint internal functors, and we do not have to use repeatedly boundless diagrammatic constructions.

Originally [5] elementary topoi were defined to be left and right exact cartesian closed categories with a subobject classifier, and consequently the semantics, and in particular its existential part, was based on these axioms [21]. We shall prove that the right exactness is a consequence of the remaining axioms, and therefore we are not allowed to apply at least the existential part of the semantics in our arguments. For this reason we shall derive directly the elementary description for most of the constructions we perform, but we shall also do so because this is the way semantics comes about. The semantics is not a logical system we have got to learn before we can do anything else.

Due to the fact that the logic of an elementary topos $\underline{\mathbf{E}}$ is not necessarily Boolean we now have the possibility of investigating the logical invariance of properties of a mathematical concept. The example in appendix 2 shows that the property that an arbitrary subobject of a finite object is itself finite is accidental, i.e. it depends on the logic in $\underline{\mathbf{E}}$. Indeed, this property is valid iff the logic in $\underline{\mathbf{E}}$ is Boolean. On the other hand, all the remaining properties usually connected with the concept of finiteness are consequences of this concept itself, i.e. they are independent of the logic in $\underline{\mathbf{E}}$ and valid in all elementary topoi.

The outcome of pointwise investigations of this kind may, as the example shows, be rather surprising, but there is still a much more dynamic question to be resolved: To what extent is a mathematical concept preserved by geometric functors on elementary topoi? In the case of local homeomorphisms, i.e. when the inverse image functor is logical, any construction based on the axioms will be preserved by the inverse image functor. Despite the trivial nature of this statement it is extremely important, allowing the good notion of combinators. As for the general case of geometric functors we shall make a detailed study of this question of invariance for internal completeness of internally ordered objects, for universal quantification, for the principle of transfinite induction and for the concept of Sierpiński-finiteness. The ideas involved in these studies are quite general and may easily be applied to other situations.

By studying a mathematical concept relative to an elementary topos $\underline{\mathbf{E}}$ we may grind the concept such that it becomes a more efficient mathematical tool. E.g. studying the concept of an atom in $\underline{\mathbf{E}}$ allows us to establish not only Stone's characterization of power objects in $\underline{\mathbf{E}}$ explaining the tripleability of $P : \underline{\mathbf{E}}^{op} \to \underline{\mathbf{E}}$, but we also see that the ground concept of atoms is now strong enough to verify that a logical functor with a right adjoint is essential, i.e. it has a left adjoint, as well as to prove that if X is a finite object in $\underline{\mathbf{E}}$, then $K(X)$ is also finite (in $\underline{\mathit{Sets}}$ this means that the set of finite subsets of a finite set is a finite set itself).

The title "Lattice Theoretic and Logical Aspects of Elementary Topoi" may be a bit misleading as it refers to the methods of proof rather than to the content itself. Indeed, what this work is actually about is an investigation of some of the naive set theoretical methods and results of the 20's and 30's studied in the context of elementary topoi, not only in order to obtain a better understanding of these ideas themselves by viewing them under the new possibility of changing the logic and the universe of discourse, but also in order to apply some of these powerful ideas of Sierpiński, Stone, Tarski and other mathematicians of these decades not only to elementary topoi one by one but also to the continuous transformation along geometric functors.

I would like to thank A. J. Kock and G. C. Wraith for explaining the subject to me in their lectures on “Elementary Toposes”, F. W. Lawvere for pointing out the possibilities of Mathematics in his lectures on ”The Foundation of Analysis”, and all three of them for their capable mathematical guidance, discussions, criticism and personal friendship.

Chapter 1

Definition and Technical Tools

An elementary topos is a left exact category $\underline{\mathbf{E}}$ with an exponentiable subobject classifier $true : 1 \rightarrowtail \Omega$.

Thus an elementary topos $\underline{\mathbf{E}}$ has a terminal object 1, binary cartesian products and equalizers.

For each object X in $\underline{\mathbf{E}}$ the map

$$in_X : \mathrm{Hom}_{\underline{\mathbf{E}}}(X, \Omega) \to P_*(X)$$

which to a morphism $f : X \to \Omega$ assigns the subobject of X which is represented by the inverse image of $true$ along f is a bijection.

The elements of the set $\mathrm{Hom}_{\underline{\mathbf{E}}}(X, \Omega)$ will be called characters, and the inverse of in_X will be denoted ch_X.

Finally, the exponentiability of the subobject classifier means that there exists a contravariant functor

$$P : \underline{\mathbf{E}}^{op} \to \underline{\mathbf{E}}$$

and a natural isomorphism

$$c_{B,A} : \mathrm{Hom}_{\underline{\mathbf{E}}}(B \times A, \Omega) \to \mathrm{Hom}_{\underline{\mathbf{E}}}(B, P(A)).$$

If $f \in \mathrm{Hom}_{\underline{\mathbf{E}}}(A, B)$ we shall think of f as a map from A to B or as an element of B defined on A, and accordingly we shall use the notation $f : A \to B$ or $f \in B$. The latter symbol suppresses the domain of definition of f but as this is always a well-defined object in $\underline{\mathbf{E}}$ the missing index will cause no ambiguity.

We begin the study of elementary topoi with a result which is due to A. Kock.

Theorem 1.1. *Any elementary topos $\underline{\mathbf{E}}$ is cartesian closed, i.e. it has exponentiation.*

Proof. First we observe that the internal power objects (i.e. the objects of the form $P(A)$) are exponentiable.

Indeed, if $A \in |\underline{\mathbf{E}}|$ then the contravariant functor

$$P_A : \underline{\mathbf{E}}^{op} \to \underline{\mathbf{E}}$$

given by $P_A(B) = P(B \times A)$ and $P_A(f) = P(f \times id_A)$ and the natural isomorphism

$$d_{C,B,A} : \mathrm{Hom}_{\underline{\mathbf{E}}}(C \times B, P(A)) \to \mathrm{Hom}_{\underline{\mathbf{E}}}(C, P_A(B))$$

determined uniquely by the following commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\underline{\mathbf{E}}}(C \times B, P(A)) & \xrightarrow{\ d\ } & \mathrm{Hom}_{\underline{\mathbf{E}}}(C, P_A(B)) \\
\uparrow{\scriptstyle c} & & \uparrow{\scriptstyle c} \\
\mathrm{Hom}_{\underline{\mathbf{E}}}((C \times B) \times A, \Omega) & \xleftarrow{\ a^*\ } & \mathrm{Hom}_{\underline{\mathbf{E}}}(C \times (B \times A), \Omega),
\end{array}$$

i.e. $d_{C,B,A} = c^{-1}_{C \times B,A} \circ {a^{-1}_{C,B,A}}^* \circ c_{C,B \times A}$ proves that the object $P(A)$ is exponentiable.

Recall the construction of $\{\ \}_A : A \to P(A)$ of the diagonal on A under the bijection:

$$P_*(A \times A) \xrightarrow{ch} \mathrm{Hom}_{\underline{\mathbf{E}}}(A \times A, \Omega) \xrightarrow{c} \mathrm{Hom}_{\underline{\mathbf{E}}}(A, P(A))$$

$$\Delta_A \longmapsto \delta_A \longmapsto \{\ \}_A$$

If $N \in |\underline{\mathbf{E}}| \quad a, b \in A \quad$ (i.e. $a, b \in \mathrm{Hom}_{\underline{\mathbf{E}}}(N, A)$) then

$a \circ \{\ \}_A = b \circ \{\ \}_A$ iff

$a \times id_A \circ \delta_A = b \times id_A \circ \delta_A$ iff

$\langle id_N, a \rangle = \langle id_N, b \rangle$ iff

$a = b$

i.e. $\{\ \}_A : A \to P(A)$ is a monomorphism.

Consider the pull back diagram

$$\begin{array}{ccc}
P(A) & \xrightarrow{\ s_A\ } & \Omega \\
\uparrow{\scriptstyle \{\ \}_A} & & \uparrow{\scriptstyle true} \\
A & \xrightarrow{\ !\ } & 1
\end{array}$$

i.e. $s_A = ch_{P(A)}(\{\ \}_A)$.

As the transformation $d^{-1}_{C,B,A} \circ s_{A*} \circ c_{C,B}$ is natural in C and B there exists, by the Yoneda lemma, a uniquely determined morphism

$$s_{B,A} : P_A(B) \to P(B)$$

such that the diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\underline{\mathbf{E}}}(C, P_A(B)) & \xrightarrow{\ s_{B,A*}\ } & \mathrm{Hom}_{\underline{\mathbf{E}}}(C, P(B)) \\
\uparrow{\scriptstyle d_{C,B,A}} & & \uparrow{\scriptstyle c_{C,B}} \\
\mathrm{Hom}_{\underline{\mathbf{E}}}(C \times B, P(A)) & \xrightarrow{\ s_{A*}\ } & \mathrm{Hom}_{\underline{\mathbf{E}}}(C \times B, \Omega),
\end{array}$$

is commutative and such that $s_{B,A}$ is natural in B, i.e. such that for all $f \in \mathrm{Hom}_{\underline{\mathbf{E}}}(B, D)$

$$\begin{array}{ccccc}
P_A(B) & \xrightarrow{\ s_{B,A}\ } & P(B) & \xleftarrow{\ \ulcorner true_B \urcorner\ } & \\
\uparrow{\scriptstyle P_A(f)} & & \uparrow{\scriptstyle P(f)} & & 1 \\
P_A(D) & \xrightarrow{\ s_{D,A}\ } & P(D) & \xleftarrow{\ \ulcorner true_D \urcorner\ } &
\end{array}$$

is a commutative diagram.

Thus if we define A^B and $\Gamma_{B,A}$ by the following pull back

$$\begin{array}{ccc} P_A(B) & \xrightarrow{s_{B,A}} & P(B) \\ \uparrow{\scriptstyle \Gamma_{B,A}} & & \uparrow{\scriptstyle \ulcorner true_B \urcorner} \\ A^B & \longrightarrow & 1 \end{array}$$

we have that the assignment $B \mapsto A^B$ extends to a subfunctor of the functor P_A, i.e. given $f \in \mathrm{Hom}_{\underline{\mathbf{E}}}(B, D)$ there exists a uniquely determined morphism $A^f : A^D \to A^B$ such that

$$\begin{array}{ccc} P_A(D) & \xrightarrow{P_A(f)} & P_A(B) \\ \uparrow{\scriptstyle \Gamma_{D,A}} & & \uparrow{\scriptstyle \Gamma_{B,A}} \\ A^D & \xrightarrow{A^f} & A^B \end{array}$$

is commutative.

Now

$f_1 : C \to A^B$ iff

$f_2 : C \to P_A(B)$ and $f_2 \circ s_{B,A} = !_C \circ \ulcorner true_B \urcorner$ iff

$f_3 : C \times B \to P(A)$ and $f_3 \circ s_A = true_{C \times B}$ iff

$f_4 : C \times B \to A$

under $f_2 = f_1 \circ \Gamma_{B,A}$ and $d^{-1}_{C,B,A}(f_2) = f_3 = f_4 \circ \{\ \}_A$, from which it follows that there exists a bijection $\underline{d}_{C,B,A}$, natural in C and B making the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\underline{\mathbf{E}}}(C \times B, P(A)) & \xrightarrow{d_{C,B,A}} & \mathrm{Hom}_{\underline{\mathbf{E}}}(C, P_A(B)) \\ \uparrow{\scriptstyle \{\ \}_{A_*}} & & \uparrow{\scriptstyle \Gamma_{B,A_*}} \\ \mathrm{Hom}_{\underline{\mathbf{E}}}(C \times B, A) & \xrightarrow{\underline{d}_{C,B,A}} & \mathrm{Hom}_{\underline{\mathbf{E}}}(C, A^B), \end{array}$$

commutative.

Thus the assignment $A \mapsto A^B$ may be extended to a covariant functor $(\ )^B$ from $\underline{\mathbf{E}}$ to $\underline{\mathbf{E}}$ such that $\underline{d}$ becomes a natural isomorphism, i.e. we have that $(\ ) \times B \dashv (\ )^B$.

This concludes the proof of Theorem 1.1. □

The cartesian adjunction

$$\underline{d}_{C,B,A} : \mathrm{Hom}_{\underline{\mathbf{E}}}(C \times B, A) \to \mathrm{Hom}_{\underline{\mathbf{E}}}(C, A^B)$$

comes in the usual way [3] with a pair of (super-) natural transformations:

$u_{C,B} : C \to (C \times B)^B$

$ev_{B,A} : A^B \times B \to A.$

Notice that the functor P is natural isomorphic to $\Omega^{(\,)}$ by the uniqueness theorem for adjoint bifunctors. Thus we may assume that $P = \Omega^{(\,)}$, but we shall keep the notation P for typographical reasons.

If $i : A \rightarrowtail X$ is a monomorphism in $\underline{\mathbf{E}}$, then i is the equalizer of $ch_X(i)$ and $true_X =!_X \circ true$ by the definition of characters. It follows that $\underline{\mathbf{E}}$ is balanced.

$\underline{\mathbf{E}}$ is well-powered as $ch_X : P_*(X) \to \mathrm{Hom}_{\underline{\mathbf{E}}}(X, \Omega)$ is an isomorphism. This means that $P*$ extends by pull backs to a contravariant functor

$$P_* : \underline{\mathbf{E}}^{op} \to \underline{Sets}.$$

Notice that this functor factors through $\underline{Ls}$, the category of lower semilattices with greatest element and left exact maps, and that P_* is representable by the subobject classifier via the natural isomorphism

(1) $\quad ch_X : P_*(X) \to \mathrm{Hom}_{\underline{\mathbf{E}}}(X, \Omega).$

From this it follows that Ω carries a uniquely determined lower semilattice structure with $true : 1 \rightarrowtail \Omega$ the greatest global section, and such that the natural isomorphism (1) lives in $\underline{Ls}$.

As the functors $(\ )^A : \underline{\mathbf{E}} \to \underline{\mathbf{E}}$ are left exact, these functors induce 2-functors on the 2-category $Ls(\underline{\mathbf{E}})$ of lower semilattice objects and left exact morphisms in $\underline{\mathbf{E}}$, and therefore the lower semilattice object $(\Omega, \wedge, true)$ in $\underline{\mathbf{E}}$ induces an $Ls(\underline{\mathbf{E}})$-structure on the internal power objects $P(A)$ in $\underline{\mathbf{E}}$. Observe that this structure $(P(A), \wedge_{P(A)}, \ulcorner true_A \urcorner)$ on $P(A)$ is determined by the fact that the natural isomorphism

(2) $\quad P_*(B \times A) \xrightarrow{ch_{B\times A}} \mathrm{Hom}_{\underline{\mathbf{E}}}(B \times A, \Omega) \xrightarrow{c_{B,A}} \mathrm{Hom}_{\underline{\mathbf{E}}}(B, P(A))$

lives in $\underline{Ls}$.

Combining the classifying property of Ω and the exponential adjointness as in (2) shows that a relation $R \rightarrowtail B \times A$ corresponds under $ch_{B\times A}$ to a character $ch(R) : B \times A \to \Omega$ which again is given via $c_{B,A}$ by a morphism $\uparrow seg_R : B \to P(A)$ (the $\uparrow$-segment of the relation R). In particular we see that

$$ch(R) =\uparrow seg_R \times id_A \circ ev_{A,\Omega}.$$

Based on these observations we define **the universal ϵ relation on A** by the following pull back diagram

$$\begin{array}{ccc} P(A) \times A & \xrightarrow{ev_{A,\Omega}} & \Omega \\ \uparrow & & \uparrow {\scriptstyle true} \\ \epsilon_A & \xrightarrow{\ !\ } & 1. \end{array}$$

The fact that the relation $R \rightarrowtail B \times A$ can be obtained as the inverse image of ϵ_A along $\uparrow seg_R \times id_A$ leads us to introduce the following terminology:

If $N \in |\underline{\mathbf{E}}|$ $\quad M \in P(A)$ $\quad$ and $\quad a \in A$ $\quad$ (i.e. $M : N \to P(A)$ and $a : N \to A$) we shall write

(3) $a \underline{\in} M$ iff $\langle M, a\rangle \circ ev_{A,\Omega} = true_N$

i.e. $\underline{\in}$ is a well defined relation on the set

$$\mathrm{Hom}_{\underline{\mathbf{E}}}(N, P(A)) \times \mathrm{Hom}_{\underline{\mathbf{E}}}(N, A).$$

In this terminology we have for a fixed relation $R \rightarrowtail B \times A$ in $\underline{\mathbf{E}}$:

$\forall N \in |\underline{\mathbf{E}}| \quad \forall b \in B \quad \forall a \in A$:

$\langle b, a\rangle$ factors through R iff

$\langle b, a\rangle \circ ch(R) = true_N$ iff

$\langle b\circ \uparrow seg_R, a\rangle \circ ev_{A,\Omega} = true_N$ iff

$a \underline{\in} b\circ \uparrow seg_R$

It is important for the understanding to observe that these relations are stable under left-composition, i.e. if $M \in |\underline{\mathbf{E}}| \quad n \in N$ and $a \underline{\in} b\circ \uparrow seg_R$ then $n \circ a \underline{\in} n \circ b\circ \uparrow seg_R$. This is an easy consequence of the universal property of pull backs.

The elements of $\mathrm{Hom}_{\underline{\mathbf{E}}}(B, P(A))$ should be thought of as B-indexed families of subobjects of A. This point of view is supported by the following:

Extensionality Principle. If $M, N \in \mathrm{Hom}_{\underline{\mathbf{E}}}(B, P(A))$ i.e. M and N are B-indexed families of subobjects of A then

$M = N$ iff M and N have the same elements, i.e.

$\forall I \in |\underline{\mathbf{E}}| \quad \forall b \in B \quad \forall a \in A : a \underline{\in} b \circ M$ iff $a \underline{\in} b \circ N$.

If $R \rightarrowtail B \times A$ is a relation in $\underline{\mathbf{E}}$, and Q is a property which makes sense in *Sets* for a binary relation, then we say that R satisfies the property Q iff $\forall N \in |\underline{\mathbf{E}}|$ the induced relation on the set $\mathrm{Hom}_{\underline{\mathbf{E}}}(N, B) \times \mathrm{Hom}_{\underline{\mathbf{E}}}(N, A)$, given by $\forall b \in B\ \forall a \in A$: bRa iff $\langle b, a\rangle$ factors through R, has the property Q.

E.g. if $R \rightarrowtail A \times A$, we say that R is reflexive iff

$\forall I \in |\underline{\mathbf{E}}| \quad \forall a \in A \ : \ a \underline{\in} a\circ \uparrow seg_R$.

We shall not bore the reader with repeating the definitions of transitivity, symmetry and antisymmetry of a relation in $\underline{\mathbf{E}}$.

An internally ordered object in $\underline{\mathbf{E}}$ is a pair $(A, \uparrow seg_R)$ where $A \in |\underline{\mathbf{E}}|$ and $\uparrow seg_R : A \to P(A)$ is the $\uparrow$-segment of a relation $R \rightarrowtail A \times A$ which is reflexive, transitive and antisymmetric.

If $(A, \uparrow seg_R)$ is an internally ordered object in $\underline{\mathbf{E}}$ we shall frequently write $\uparrow seg_A = \uparrow seg_R$ and use the more suggestive notation

$a \geq b$ iff aRb iff $a \underline{\in} b\circ \uparrow seg_A$ $(= a \underline{\in} b\circ \uparrow seg_R)$, where a and b are elements of A defined on N, say.

Any lower semilattice object $(A, \wedge)$ in $\underline{\mathbf{E}}$ carries a canonical internal order relation which is constructed by the following equalizer diagram:

$$R \rightarrowtail A \times A \underset{pr_0}{\overset{\wedge}{\rightrightarrows}} A$$

i.e. $\forall N \in |\underline{\mathbf{E}}| \quad \forall a, b \in A : a \leqslant b \quad \text{iff} \quad a = a \wedge b.$

By applying this to the internal power objects in $\underline{\mathbf{E}}$ we see that the lower semilattice structure defined by the natural isomorphism (2) defines an internal ordering (the canonical ordering) on the objects $P(A)$. We shall use the notation $(P(A), \uparrow seg_{P(A)})$ for this ordering on $P(A)$.

Let us record the defining property of the canonical ordering on $P(A)$:

If $B \in |\underline{\mathbf{E}}| \quad M, N \in P(A)$ then $M \leqslant N$ iff

$\forall I \in |\underline{\mathbf{E}}| \quad \forall b \in B \quad \forall a \in A \ : \ a \underline{\in} b \circ M$ implies $a \underline{\in} b \circ N$

Notice that the extensionality principle is stating that the canonical ordering $\uparrow seg_{P(A)}$ on $P(A)$ is antisymmetric.

If $f \in \mathrm{Hom}_{\underline{\mathbf{E}}}(A, C) \quad I \in \underline{\mathbf{E}} \quad M \in P(C)$ and $a \in A$ then

(4) $\quad a \underline{\in} M \circ P(f) \quad$ iff $\quad a \circ f \underline{\in} M.$

This rule is one of the most important features of $P(f)$ (the internal substitution along f). The proof is a direct translation of the supernaturality of ev :

$$P(f) \times id_A \circ ev_{A,\Omega} = id_{P(C)} \times f \circ ev_{C,\Omega}.$$

If $(A, \uparrow seg_A)$ and $(B, \uparrow seg_B)$ are internally ordered objects in $\underline{\mathbf{E}}$, a morphism $f \in \mathrm{Hom}_{\underline{\mathbf{E}}}(A, B)$ will be called an internal functor from A to B provided $\forall N \in |\underline{\mathbf{E}}|$ the map

$$\mathrm{Hom}_{\underline{\mathbf{E}}}(N, f) : \mathrm{Hom}_{\underline{\mathbf{E}}}(N, A) \to \mathrm{Hom}_{\underline{\mathbf{E}}}(N, B)$$

is order-preserving, i.e.

$\forall N \in |\underline{\mathbf{E}}| \quad \forall x, y \in A \ : \ x \leqslant y$ implies $x \circ f \leqslant y \circ f.$

Notice that we can express that f is an internal functor by the following inequality

$$\uparrow seg_A \leqslant f \circ \uparrow seg_B \circ P(f)$$

Notice that the internal substitution along a morphism is an internal functor.

Let $Ord(\underline{\mathbf{E}})$ be the 2-category of internally ordered objects in $\underline{\mathbf{E}}$ and internal functors. (The 2-structure comes from the order on the hom sets).

If $R \rightarrowtail B \times A$ is a relation in $\underline{\mathbf{E}}$ we may consider the inverse relation $R^{-1} \rightarrowtail A \times B = R \rightarrowtail B \times A \underset{\sim}{\to} A \times B$, where the isomorphism is the cartesian twist.

The morphism $\uparrow seg_{R^{-1}}$ is denoted $\downarrow seg_R$ (the $\downarrow$-segment of the relation R).

$\forall N \in |\underline{\mathbf{E}}| \quad \forall b \in B \quad \forall a \in A \quad :$

(5) $$b \mathrel{\underline{\in}} a \circ \downarrow seg_R \qquad \text{iff} \qquad a \mathrel{\underline{\in}} b \circ \uparrow seg_R.$$

If $R \rightarrowtail A \times A$ then R is symmetric iff $\downarrow seg_R = \uparrow seg_R$. Likewise R is reflexive iff $\{\ \}_A \leqslant \uparrow seg_R$ iff $\{\ \}_A \leqslant \downarrow seg_R$. Also, R is transitive iff one of the four equivalent inequalities is satisfied:

i) $\downarrow seg_R \leqslant \downarrow seg_R \circ \downarrow seg_{P(A)} \circ P(\downarrow seg_R)$

ii) $\downarrow seg_R \leqslant \uparrow seg_R \circ \uparrow seg_{P(A)} \circ P(\uparrow seg_R)$

iii) $\uparrow seg_R \leqslant \downarrow seg_R \circ \uparrow seg_{P(A)} \circ P(\downarrow seg_R)$

iv) $\uparrow seg_R \leqslant \uparrow seg_R \circ \downarrow seg_{P(A)} \circ P(\uparrow seg_R)$

If R is reflexive and transitive then i) - iv) are equalities. In this case R is antisymmetric iff $\downarrow seg_R$ is monic iff $\uparrow seg_R$ is monic.

The involution $\uparrow seg_R \longleftrightarrow \downarrow seg_R$ makes it possible to introduce the notion of contravariant internal functors on internally ordered objects in $\underline{\mathbf{E}}$. If $(A, \uparrow seg_A)$ and $(B, \uparrow seg_B)$ are in $Ord(\underline{\mathbf{E}})$, a morphism $g \in \mathrm{Hom}_{\underline{\mathbf{E}}}(A, B)$ will be called a contravariant internal functor from A to B provided $\forall N \in |\underline{\mathbf{E}}|$ the map

$$\mathrm{Hom}_{\underline{\mathbf{E}}}(N, g) : \mathrm{Hom}_{\underline{\mathbf{E}}}(N, A) \to \mathrm{Hom}_{\underline{\mathbf{E}}}(N, B)$$

is order reversing, i.e.

$$\forall N \in |\underline{\mathbf{E}}| \quad \forall x, y \in A \quad : \quad x \leqslant y \quad \text{implies} \quad y \circ g \leqslant x \circ g.$$

or equivalently

$$\downarrow seg_A \leqslant g \circ \uparrow seg_B \circ P(g)$$

Notice that if $(A, \uparrow seg_A) \in |Ord(\underline{\mathbf{E}})|$ then $\downarrow seg_A$ is an internal functor from $(A, \uparrow seg_A)$ to $(P(A), \uparrow seg_{P(A)})$, and $\uparrow seg_A$ is a contravariant internal functor on the same objects.

Any 2-category admits the theory of adjoint 1-cells. For $Ord(\underline{\mathbf{E}})$ we shall use the following terminology. If

(6) $$(A, \uparrow seg_A) \underset{g}{\overset{f}{\rightleftarrows}} (B, \uparrow seg_B))$$

is a diagram in $Ord(\underline{\mathbf{E}})$ then

$f \dashv g$ iff

$\forall N \in |\underline{\mathbf{E}}| : \mathrm{Hom}_{\underline{\mathbf{E}}}(N, f) \dashv \mathrm{Hom}_{\underline{\mathbf{E}}}(N, g)$ iff

$\forall N \in |\underline{\mathbf{E}}| \quad \forall a \in A \quad \forall b \in B : a \circ f \leqslant b$ iff $a \leqslant b \circ g$ iff

$f \circ \uparrow seg_B = \uparrow seg_A \circ P(g)$ iff

$\downarrow seg_B \circ P(f) = g \circ \downarrow seg_A$ iff

$id_A \leqslant f \circ g$ and $g \circ f \leqslant id_B$.

Adjoint internal functors determine each other uniquely as the order relations are antisymmetric. If $f \dashv g$ then $f = f \circ g \circ f$ and $g = g \circ f \circ g$. It follows that if $f \dashv g$ then

f is monic iff g is epic iff $id_A = f \circ g$

and

f is epic iff g is monic iff $g \circ f = id_B$.

By dualizing we get the notion of contravariant internal functors adjoint on the right. If in

$$(A, \uparrow seg_A) \underset{s}{\overset{r}{\rightleftarrows}} (B, \uparrow seg_B)) \tag{7}$$

r and s are contravariant internal functors then

$r \perp s$ iff

$\forall N \in |\underline{\mathbf{E}}| : \mathrm{Hom}_{\underline{\mathbf{E}}}(N, r) \perp \mathrm{Hom}_{\underline{\mathbf{E}}}(N, s)$ iff

$\forall N \in |\underline{\mathbf{E}}| \quad \forall a \in A \quad \forall b \in B : b \leqslant a \circ r$ iff $a \leqslant b \circ s$ iff

$s \circ \downarrow seg_A = \uparrow seg_B \circ P(r)$ iff

$r \circ \downarrow seg_B = \uparrow seg_A \circ P(s)$ iff

$id_A \leqslant r \circ s$ and $id_B \leqslant s \circ r$

etc.

The composition of adjoints and of adjoints on the right follows the classical rules.

The category $Ord(\underline{\mathbf{E}})$ has finite inverse limits and the forgetful functor from $Ord(\underline{\mathbf{E}})$ to $\underline{\mathbf{E}}$ preserves them.

$(1, \ulcorner true \urcorner)$ is the terminal object in $Ord(\underline{\mathbf{E}})$. Notice that any global section in an internally ordered object in $\underline{\mathbf{E}}$ is automatically an internal functor.

If

$$K \overset{i}{\rightarrowtail} A \underset{g}{\overset{f}{\rightrightarrows}} B$$

is the underlying equalizer diagram of a pair of internal functors f and g, then

$$\uparrow seg_K = i \circ \uparrow seg_A \circ P(i)$$

equips K with the structure of an internally ordered object such that

$$(K, \uparrow seg_K) \overset{i}{\rightarrowtail} (A, \uparrow seg_A) \underset{g}{\overset{f}{\rightrightarrows}} (B, \uparrow seg_B)$$

is an equalizer diagram in $Ord(\underline{\mathbf{E}})$.

If $A, B \in |\underline{\mathbf{E}}|$ consider the morphism

(8) $\tilde{p}_{A,B} : P(A) \times P(B) \xrightarrow[P(p_0)\times P(p_1)]{} P(A \times B) \times P(A \times B) \xrightarrow[\wedge_{P(A\times B)}]{} P(A \times B)$

which is uniquely determined by the rule:

$\forall N \in |\underline{\mathbf{E}}| \quad \forall S \in P(A) \quad \forall T \in P(B) \quad \forall a \in A \quad \forall b \in B :$

$\langle a, b\rangle \,\underline{\in}\, \langle S, T\rangle \circ \ \tilde{p}_{A,B}$ iff $a \,\underline{\in}\, S$ and $b \,\underline{\in}\, T$.

If $(A, \uparrow seg_A)$ and $(B, \uparrow seg_B)$ are in $Ord(\underline{\mathbf{E}})$ then

$$\uparrow seg_{A\times B} = \uparrow seg_A \times \uparrow seg_B \circ \tilde{p}_{A,B}$$

is the order relation on $A \times B$ which makes

$$(A, \uparrow seg_A) \xleftarrow[p_0]{} (A \times B, \uparrow seg_{A\times B}) \xrightarrow[p_1]{} (B, \uparrow seg_B)$$

a cartesian product diagram in $Ord(\underline{\mathbf{E}})$.

If $(A, \uparrow seg_A)$, $(B, \uparrow seg_B)$ and $(C, \uparrow seg_C)$ are in $Ord(\underline{\mathbf{E}})$ and if $\circ : A \times B \longrightarrow C$ is an internal (bi-) functor, we say that $\circ$ admits an exponential $\to$ ($\circ$ expo $\to$) iff there exists an internal (bi-) functor $\to: B^{op} \times C \longrightarrow A$ such that

$\forall N \in |\underline{\mathbf{E}}| \quad \forall a \in A \quad \forall b \in B \quad \forall c \in C \quad :$

(9) $a \circ b \leqslant c$ iff $a \leqslant b \to c$

Clearly, $\circ$ and $\to$ determine each other uniquely.

Proposition 1.1. *Let* $\Rightarrow: \Omega \times \Omega \longrightarrow \Omega$ *be the exponential adjoint of* $\uparrow seg_\Omega : \Omega \to P(\Omega)$*, then* $\Rightarrow$ *is an internal (bi-) functor and* $\wedge$ *expo* $\Rightarrow$*. (*$\Rightarrow$ *is called the implication on* Ω*).*

We leave the proof of this proposition to the reader as it can be found in any treatment of elementary topoi, but we shall give the proof of the following proposition which equips Ω with a binary union.

The idea is the following. As $(\Omega, \wedge, true, \Rightarrow)$ is a Heyting algebra object in $\underline{\mathbf{E}}$, if it has a binary union $\vee : \Omega \times \Omega \to \Omega$ this union must satisfy the equation

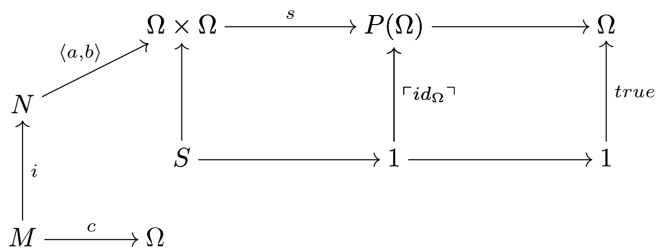
(10) $(a \vee b) \Rightarrow c \quad = \quad (a \Rightarrow c) \wedge (b \Rightarrow c)$

for all elements in Ω with domain N, $N \in |\underline{\mathbf{E}}|$.

Let $s : \Omega \times \Omega \to P(\Omega)$ be the exponential adjoint of the morphism

$$(\Omega \times \Omega) \times \Omega \xrightarrow{id\times\Delta} (\Omega \times \Omega) \times (\Omega \times \Omega) \xrightarrow{m} (\Omega \times \Omega) \times (\Omega \times \Omega) \xrightarrow{\Rightarrow\times\Rightarrow} \Omega \times \Omega \xrightarrow{\wedge} \Omega$$

where m is the middle four interchange [3], and let $\vee$ be the upper morphism in the pull back diagram:



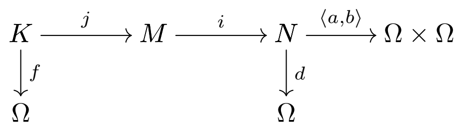
By the construction of $\vee$ we have that

$\forall N \in |\underline{\mathbf{E}}| \quad \forall a, b \in \Omega \quad a \vee b = true_N$ iff

$\forall M \in |\underline{\mathbf{E}}| \quad \forall c \in \Omega \ : \ (i \circ a \Rightarrow c) \wedge (i \circ b \Rightarrow c) = c$

Proposition 1.2. *The operation $\vee : \Omega \times \Omega \to \Omega$ is a binary union on Ω.*

Proof.



If $N \in |\underline{\mathbf{E}}| \quad \forall a, b, d \in \Omega$ and $a \leqslant d$ and $b \leqslant d$ then $a \vee b \leqslant d$. Indeed, if $M \in |\underline{\mathbf{E}}| \quad \forall i \in N$ and $i \circ a \vee i \circ b = true_M$, then

$$i \circ d = (i \circ a \Rightarrow i \circ d) \wedge (i \circ b \Rightarrow i \circ d) = true_M \wedge true_M = true_M,$$

it follows that $a \vee b \leqslant d$.

On the other hand, $a \leqslant a \vee b$. Indeed, if $M \in |\underline{\mathbf{E}}| \quad i \in N$ and $i \circ a = true_M$, then $\forall K \in |\underline{\mathbf{E}}| \quad \forall j \in M \quad \forall f \in \Omega \ :$

$$(j \circ i \circ a \Rightarrow f) \wedge (j \circ i \circ b \Rightarrow f) = (true_K \Rightarrow f) \wedge (j \circ i \circ b \Rightarrow f) = f \wedge (j \circ i \circ b \Rightarrow f) = f,$$

whence $i \circ a \vee i \circ b = true_M$, and so $a \leqslant a \vee b$. Dually $b \leqslant a \vee b$. It follows that (10) is generally valid.

This concludes the proof of Proposition 1.2. □

Using the fact that the functors $(\ )^A$ are left exact we see that the internal power objects are equipped with an implication $\Rightarrow_{P(A)}$ and a binary union $\vee_{P(A)}$.

Proposition 1.3. $\forall A \in |\underline{\mathbf{E}}| \ : \downarrow\!seg_A \circ P(\{\ \}_A) = id_{P(A)}$.

Proof. $\forall I \in \underline{\mathbf{E}} \quad \forall M \in P(A) \quad \forall a \in A :$

$a \mathrel{\underline{\in}} M \circ \downarrow\!seg_P(A) \circ P(\{\ \}_A)$ iff

$a \circ \{\ \}_A \mathrel{\underline{\in}} M \circ \downarrow\!seg_P(A)$ iff

$a \circ \{\ \}_A \leqslant M$ iff

$a \mathrel{\underline{\in}} M$

i.e. $\downarrow\!seg_A \circ P(\{\ \}_A) = id_{P(A)}$.

□

Proposition 1.3 should be viewed as an internal form of the extensionality principle.

Let $(A, \uparrow seg_A) \in |Ord(\underline{\mathbf{E}})|$. we shall say that A is internally co-complete iff the internal functor $\downarrow seg_A : A \to P(A)$ has a left adjoint, denoted $sup_A : P(A) \to A$ (if it exists). sup_A is characterized by any of the two equivalent equations

i) $sup_A \circ \uparrow seg_A = \uparrow seg_{P(A)} \circ P(\downarrow seg_A)$

ii) $\downarrow seg_A \circ P(sup_A) = \downarrow seg_A \circ \downarrow seg_{P(A)}$

as well as by the elementary description :

$$\forall N \in |\underline{\mathbf{E}}| \quad \forall M \in P(A) \quad \forall a \in A \quad :$$

(11) $\quad M \circ sup_A \leqslant a \qquad \text{iff} \qquad M \leqslant a \circ \downarrow seg_A$

Dually, $(A, \uparrow seg_A)$ is said to be internally complete iff the contravariant internal functor $\uparrow seg_A : A \to P(A)$ has an adjoint on the right, denoted $inf_A : P(A) \to A$ (if it exists). Again, inf_A is characterized by

i) $inf_A \circ \downarrow seg_A = \uparrow seg_{P(A)} \circ P(\uparrow seg_A)$

ii) $\uparrow seg_A \circ \downarrow seg_{P(A)} = \uparrow seg_A \circ P(inf_A)$

and by the elementary description :

$$\forall N \in |\underline{\mathbf{E}}| \quad \forall M \in P(A) \quad \forall a \in A \quad :$$

(12) $\quad a \leqslant M \circ inf_A \qquad \text{iff} \qquad M \leqslant a \circ \uparrow seg_A$

Let us finish this chapter with a few remarks on the functor

$$P : \underline{\mathbf{E}}^{op} \longrightarrow \underline{\mathbf{E}}.$$

As $P = \Omega^{(\,)}$ we know that P is adjoint to itself on the right.

If we consider P as a covariant functor we have that

(13) $\quad \underline{\mathbf{E}} \xrightarrow{P} \underline{\mathbf{E}}^{op} \dashv \underline{\mathbf{E}}^{op} \xrightarrow{P} \underline{\mathbf{E}}$

under the natural isomorphism

$$\mathrm{Hom}_{\underline{\mathbf{E}}}(B, P(A)) \underset{\sim}{\longrightarrow} \mathrm{Hom}_{\underline{\mathbf{E}}}(A, P(B))$$

which is induced by the natural isomorphism c and the cartesian twist.

The unity for this adjoint situation is $w_A : A \to PP(A)$ which is defined by the following rule:

$$\forall N \in |\underline{\mathbf{E}}| \quad \forall M \in P(A) \quad \forall a \in A \quad : \quad M \underset{\sim}{\in} a \circ w_A \qquad \text{iff} \qquad a \underset{\sim}{\in} M$$

i.e. $w_A = \{\ \}_A \circ \uparrow seg_{P(A)}$ (the principal ultrafilters on A).

The monad on $\underline{\mathbf{E}}$ generated by (13) is called the double dualization monad (with respect to Ω) on $\underline{\mathbf{E}}$. Notice that the multiplication m is given by $m_A = P(w_A)$.

Observe that w is pointwise monic. Indeed $\{\ \}_A$ is monic by the proof of Theorem 1.1, and $\uparrow seg_{P(A)}$ is monic due to the antisymmetry of the canonical ordering on $P(A)$.

Chapter 2

Internal Completeness in Elementary Topoi

In this chapter let $\underline{\mathbf{E}}$ be a fixed elementary topos. In the first chapter we saw that the internal power objects $P(A)$ in $\underline{\mathbf{E}}$ had the structure $(P(A), \wedge_{P(A)}, \ulcorner true_A \urcorner, \Rightarrow_{P(A)})$ of a Heyting algebra object, and that the internal substitution morphisms $P(f)$ were internal functors. Our first aim is to construct adjoints to these internal functors.

Proposition 2.1. *Let $f \in \mathrm{Hom}_{\underline{\mathbf{E}}}(A, B)$ then the internal functor $P(f)$ has a right adjoint $\forall_f$ (the internal universal quantification along f).*

$$\forall_f = P(A) \xrightarrow{\downarrow seg_{P(A)}} PP(A) \xrightarrow{PP(f)} PP(B) \xrightarrow{P(\{\ \}_B)} P(B).$$

Proof. $\forall_f$ is an internal functor by construction, and

i) $P(f) \circ \forall_f = P(f) \circ \downarrow seg_{P(A)} \circ PP(f) \circ P(\{\ \}_B) \geqslant$

$\downarrow seg_{P(B)} \circ P(\{\ \}_B) = id_{P(B)}$ and

ii) $\forall_f \circ P(f) = \downarrow seg_{P(A)} \circ PP(f) \circ P(\{\ \}_B) \circ P(f) =$

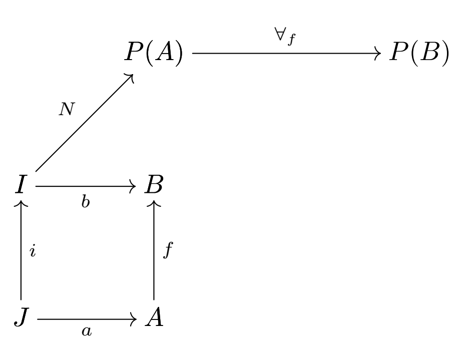
$\downarrow seg_{P(A)} \circ P(f \circ \{\ \}_B \circ P(f)) \leqslant \downarrow seg_{P(A)} \circ P(\{\ \}_A) = id_{P(A)}$.

This proves that $P(f) \dashv \forall_f$. ☐

Elementary description of $\forall_f$:

$\forall I \in |\underline{\mathbf{E}}| \quad \forall N \in P(A) \quad \forall b \in B: \; b \underline{\in} N \circ \forall_f$ iff

$\forall J \in |\underline{\mathbf{E}}| \quad \forall i \in I \quad \forall a \in A: \; a \circ f = i \circ b$ implies $a \underline{\in} i \circ N$



Proof. $b \underline{\in} N \circ \forall_f$ iff $b \circ \{\ \}_B \leqslant N \circ \forall f$ iff $b \circ \{\ \}_B \circ P(f) \leqslant N$. ☐

Corollary 2.1. *$\underline{\mathbf{E}}$ has a strict initial object $\varnothing$.*

Proof. Consider the global section $false : 1 \to \Omega$ defined by the following equation:

$$\ulcorner false \urcorner = \ulcorner id_\Omega \urcorner \circ \forall_{!_\Omega}$$

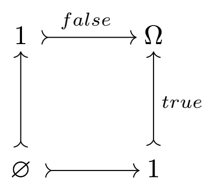
We claim that $false \dashv !_\Omega$. Now,

$$\ulcorner false_\Omega \urcorner = \ulcorner !_\Omega \circ false \urcorner = \ulcorner false \urcorner \circ P(!_\Omega) =$$

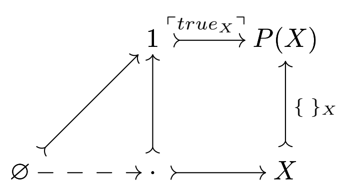
$$\ulcorner id_\Omega \urcorner \circ \forall_{!_\Omega} \circ P(!_\Omega) \leqslant \ulcorner id_\Omega \urcorner$$

and consequently $false \dashv !_\Omega$.

Consider the following pull back diagram



By the uniqueness theorem for adjoint internal functors we have that $\varnothing$ is the smallest subobject of 1, and this implies that $\varnothing$ satisfies the uniqueness property of an initial object as $\underline{\mathbf{E}}$ has equalizers. Accordingly, the next diagram shows that $\varnothing$ is an initial object.



Finally, $\varnothing$ is a strict initial object as $\underline{\mathbf{E}}$ is cartesian closed. ☐

Corollary 2.2. *If $f \in \mathrm{Hom}_{\underline{\mathbf{E}}}(A, B)$ then f is monic iff $\{\ \}_A = f \circ \{\ \}_B \circ P(f)$ iff $\forall_f \circ P(f) = id_{P(A)}$ iff $P(f)$ is epic iff $\forall_f$ is monic.*

Proof. This follows from ii) in the proof of Proposition 2.1. ☐

Corollary 2.3. *The assignment $X \longmapsto P(X)$, $f \longmapsto \forall_f$ defines a functor (covariant and faithful)*

$$\forall : \underline{\mathbf{E}} \longrightarrow \underline{\mathbf{E}}$$

called internal universal quantification.

Proof. This follows from the uniqueness theorem for adjoint internal functors and the fact that P is a faithful and contravariant functor. ☐

In order to establish that $P(f)$ has a left adjoint we need the fact that the internal power objects are internally complete.

Proposition 2.2. *Let $X \in |\underline{\mathbf{E}}|$ and let $\bigcap_X$ be the internal intersection on $P(X)$ defined as follows*

$$\bigcap_X = PP(X) \xrightarrow{\uparrow seg_{PP(X)}} PPP(X) \xrightarrow{P(\uparrow seg_{P(X)})} PP(X) \xrightarrow{P(\{\ \}_X)} P(X)$$

then $\bigcap_X$ is a contravariant internal functor, $\bigcap_X \perp \uparrow seg_{P(X)}$ and furthermore $\{\ \}_{P(X)} \circ \bigcap_X = id_{P(X)}$.

Proof. $\bigcap_X$ is a contravariant internal functor by construction, and

i) $\{\ \}_{P(X)} \circ \bigcap_X = w_{P(X)} \circ P(w_{P(X)}) = id_{P(X)}$

(i.e. the unite law for the double dualization monad on $\underline{\mathbf{E}}$)

ii) $\uparrow seg_{P(X)} \circ \bigcap_X = \uparrow seg_{P(X)} \circ \uparrow seg_{PP(X)} \circ P(\uparrow seg_{P(X)}) \circ P(\{\ \}_X) =$

$\downarrow seg_{P(X)} \circ P(\{\ \}_X) = id_{P(X)}$

iii) $\forall I \in |\underline{\mathbf{E}}| \quad \forall N \in PP(X) \quad \forall A \in P(X):$

$A \in N$ iff

$A \circ \{\ \}_{P(X)} \leqslant N$ implies

$N \circ \bigcap_X \leqslant A \circ \{\ \}_{P(X)} \circ \bigcap_X = A$ iff

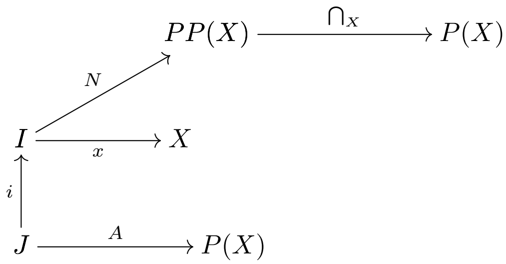
$A \in N \circ \bigcap_X \circ \uparrow seg_{P(X)}$.

I.e. $id_{PP(X)} \leqslant \bigcap_X \circ \uparrow seg_{P(X)}$, whence $\bigcap_X \perp \uparrow seg_{P(X)}$. □

Elementary description of $\bigcap_X$:

$\forall I \in |\underline{\mathbf{E}}| \quad \forall N \in PP(X) \quad \forall x \in X : x \in N \circ \bigcap_X$ iff

$\forall J \in |\underline{\mathbf{E}}| \quad \forall i \in I \quad \forall A \in P(X) : A \in i \circ N$ implies $i \circ x \in A$.



Proof. $x \in N \circ \bigcap_X$ iff $x \circ \{\ \}_X \leqslant N \circ \bigcap_X$ iff $N \leqslant x \circ \{\ \}_X \circ \uparrow seg_{P(X)}$. □

Proposition 2.3. *Let $f \in \mathrm{Hom}_{\underline{\mathbf{E}}}(A, B)$ then the internal functor $P(f)$ has a left adjoint $\exists_f$ (the internal existential quantification along f).*

$$\exists_f = P(A) \xrightarrow{\uparrow seg_{P(A)}} PP(A) \xrightarrow{PP(f)} PP(B) \xrightarrow{\bigcap_B} P(B)$$

Proof. $\exists_f$ is an internal functor by construction, and

i) $\exists_f \circ P(f) = \uparrow seg_{P(A)} \circ PP(f) \circ \bigcap_B \circ P(f) =$

$\uparrow seg_{P(A)} \circ PP(f) \circ \uparrow seg_{PP(B)} \circ P(w_B) \circ P(f) =$

$\uparrow seg_{P(A)} \circ PP(f) \circ \uparrow seg_{PP(B)} \circ PPP(f) \circ P(w_A) \geqslant$

$\uparrow seg_{P(A)} \circ \uparrow seg_{PP(A)} \circ P(w_A) = \uparrow seg_{P(A)} \circ \bigcap_A = id_{P(A)}$

ii) $P(f) \circ \exists_f = P(f) \circ \uparrow seg_{P(A)} \circ PP(f) \circ \bigcap_B \leqslant$

$\uparrow seg_{P(B)} \circ \bigcap_B = id_{P(B)}$

This proves that $\exists_f \dashv P(f)$ □

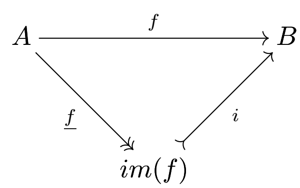
Corollary 2.4. $\underline{\mathbf{E}}$ *has epi-mono-factorization.*

Proof. Let $f \in \mathrm{Hom}_{\underline{\mathbf{E}}}(A, B)$. If $i : C \rightarrowtail B$ is any monomorphism in $\underline{\mathbf{E}}$ then f factors through i iff $f \circ ch_B(i) = true_A$ iff $\ulcorner true_A \urcorner \leqslant \ulcorner ch_B(i) \urcorner \circ P(f)$ iff $\ulcorner true_A \urcorner \circ \exists_f \leqslant \ulcorner ch_B(i) \urcorner$.

Thus if we define the image of f by the equation

$$\ulcorner ch_B(im(f)) \urcorner = \ulcorner true_A \urcorner \circ \exists_f$$

we have that $i = im(f) \rightarrowtail B$ is the smallest subobject of B through which f factors. Also the factorization



the morphism $\underline{f}$ is epic as $\underline{\mathbf{E}}$ has equalizers.

The above factorization is unique as $\underline{\mathbf{E}}$ is balanced. □

Corollary 2.5. *If* $f \in \mathrm{Hom}_{\underline{\mathbf{E}}}(A, B)$ *then* f *is epic iff* $\ulcorner true_A \urcorner \circ \exists_f = \ulcorner true_B \urcorner$ *iff* $P(f) \circ \exists_f = id_{P(B)}$ *iff* $P(f)$ *is monic iff* $\exists_f$ *is epic.*

Corollary 2.6. *The assignment* $X \longmapsto P(X)$, $f \longmapsto \exists_f$ *defines a functor (covariant and faithful)*

$$\exists : \underline{\mathbf{E}} \longrightarrow \underline{\mathbf{E}}$$

called internal existential quantification, and

$$\{\ \} = \{\{\ \}_X : X \rightarrowtail P(X)\}_{X \in |\underline{\mathbf{E}}|} : id_{\underline{\mathbf{E}}} \Longrightarrow \exists$$

is a pointwise monic natural transformation.

Proof. As P is a contravariant functor it follows from the uniqueness theorem for adjoint internal functors that $\exists$ is a well defined functor. If $f \in \mathrm{Hom}_{\underline{\mathbf{E}}}(A, B)$ then

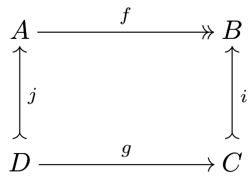
$$\{\ \}_A \circ \exists_f = \{\ \}_A \circ \uparrow seg_{P(A)} \circ PP(f) \circ \bigcap_B = w_A \circ PP(f) \circ \bigcap_B =$$

$$f \circ w_B \circ \bigcap_B = f \circ \{\ \}_B \circ \uparrow seg_{P(B)} \circ \bigcap_B = f \circ \{\ \}_A,$$

which proves that $\{\ \}$ is a natural transformation, and as the values of $\{\ \}$ are monic we know that $\exists$ is a faithful functor. □

Proposition 2.4. *In $\underline{\mathbf{E}}$ epimorphisms are preserved by pull backs.*

Proof. Let $f \in \mathrm{Hom}_{\underline{\mathbf{E}}}(A, B)$ be epic and $i \in \mathrm{Hom}_{\underline{\mathbf{E}}}(C, B)$ be monic and consider the pull back



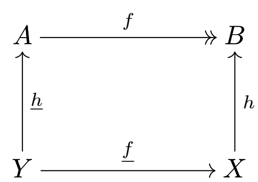
Now $\ulcorner ch_B(i) \urcorner = \ulcorner ch_B(i) \urcorner \circ P(f) \circ \exists_f = \ulcorner f \circ ch_B(i) \urcorner \circ \exists_f =$

$\ulcorner ch_A(j) \urcorner \circ \exists_f = \ulcorner ch_A(im(j)) \urcorner \circ \exists_f = \ulcorner true_D \urcorner \circ \exists_j \circ \exists_f =$

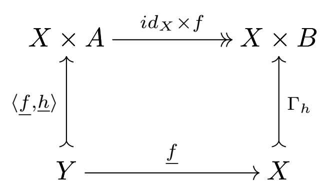
$\ulcorner true_D \urcorner \circ \exists_{j \circ f} = \ulcorner ch_B(im(j \circ f)) \urcorner$.

This proves that epimorphisms are preserved by pull backs along monomorphisms.

In order to obtain the proof in the general case we take $h \in \mathrm{Hom}_{\underline{\mathbf{E}}}(X, B)$ and consider the pull back diagram



as well as the diagram

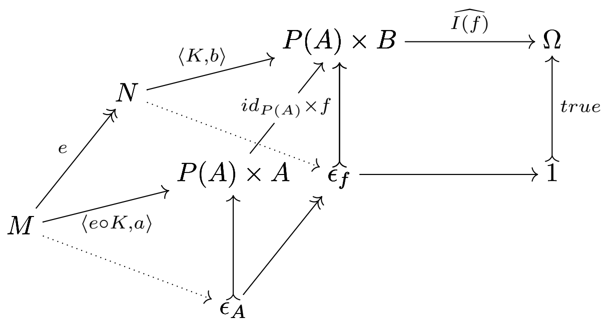


As f is epic we have that $id_X \times f$ is epic as $\underline{\mathbf{E}}$ is cartesian closed. And as the second square is a pull back iff the first one is, and as Γ_h is mono (split) it follows from the first part of the proof that $\underline{f}$ is an epimorphism.

This concludes the proof of Proposition 2.4 □

Elementary description of $\exists_f$:

If $f \in \mathrm{Hom}_{\underline{\mathbf{E}}}(A, B)$ let us consider the "classical" construction of $\exists_f$. For this clause only let us use the notation $I(f)$. We indicate the construction by the following diagram.



i.e. $\widehat{I(f)}$ classifies the image ϵ_f of ϵ_A along $id_{P(A)} \times f$.

The construction of $I(f)$ is possible due to Corollary 2.4, and by Proposition 2.4 we know that $I(f)$ has the following elementary description:

$\forall N \in |\underline{\mathbf{E}}| \quad \forall K \in P(A) \quad \forall b \in B : b \underline{\in} K \circ I(f)$ iff

$\exists M \in |\underline{\mathbf{E}}| \quad \exists e \in N$ (epi) $\exists a \in A$ such that

$$a \circ f = e \circ b \qquad \textit{and} \qquad a \underline{\in} e \circ K$$

i) $I(f)$ is an internal functor.

If $N \in |\underline{\mathbf{E}}| \quad K, L \in P(A) \quad b \in B \quad b \underline{\in} K \circ I(f)$ and $K \leqslant L$, then

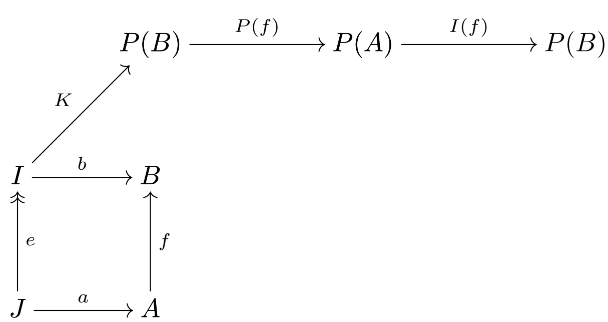
$\exists M \in |\underline{\mathbf{E}}| \quad \exists e \in N$ (epi) $\exists a \in A$ such that $a \circ f = e \circ b$

and $a \underline{\in} e \circ K$, but $K \leqslant L$, thus $e \circ K \leqslant e \circ L$ and so $a \underline{\in} e \circ L$

ii) $id_{P(A)} \leqslant I(f) \circ P(f)$.

This follows from the defining diagram of $I(f)$.

iii)



Let $I \in |\underline{\mathbf{E}}| \quad K \in P(B) \quad b \in B$ and assume that $b \underline{\in} K \circ P(f) \circ I(f)$.

Thus $\exists J \in |\underline{\mathbf{E}}| \quad \exists e \in I$ (epi) $\exists a \in A$ such that $a \circ f = e \circ b$ and

$a \underline{\in} e \circ K \circ P(f)$, i.e. $a \circ f \underline{\in} e \circ K$.

But then $e \circ b \underline{\in} e \circ K$ and so $b \underline{\in} K$ as e is epic.

iv) Summing up we see that $I(f)$ is an internal functor and that $I(f) \dashv P(f)$. It follows from the uniqueness theorem for adjoint internal functors that $I(f) = \exists_f$ as we have that both

$$I(f) \dashv P(f) \qquad and \qquad \exists_f \dashv P(f).$$

Thus the construction in Proposition 2.3 makes possible the "classical" construction of internal existential quantification and the two constructions agree. It follows that $I(f)$ and $\exists_f$ have the same elementary description.

Recall that in establishing the existence of $\exists_f$ we had to verify that the internal power objects were internally complete. This fact leads to a rapid development of the internal structure of $\underline{\mathbf{E}}$.

Let $\underline{C}(\underline{\mathbf{E}})$ be the category of internally complete ordered objects in $\underline{\mathbf{E}}$ and inf-preserving morphisms. If A and B are in $\underline{C}(\underline{\mathbf{E}})$ and $f \in \mathrm{Hom}_{\underline{\mathbf{E}}}(A, B)$ then f is inf-preserving iff the following diagram is commutative

$$\begin{array}{ccc} P(A) & \xrightarrow{\exists_f} & P(B) \\ \downarrow{\scriptstyle inf_A} & & \downarrow{\scriptstyle inf_B} \\ A & \xrightarrow{f} & B \end{array}$$

Notice that inf-preserving morphisms are automatically internal functors as

$$f = \uparrow seg_A \circ \inf_A \circ f = \uparrow seg_A \circ \exists_f \circ \inf_B.$$

Dually, let $\overline{C}(\underline{\mathbf{E}})$ be the category of internally co-complete ordered objects in $\underline{\mathbf{E}}$ and sup-preserving morphisms. Etc.

Proposition 2.5. *Internally complete ordered objects are also internally co-complete. Explicitly, if $A = (A, \uparrow seg_A, inf_A)$ is in $\underline{C}(\underline{\mathbf{E}})$ then*

$$sup_A = P(A) \xrightarrow{\exists_{\uparrow seg_A}} PP(A) \xrightarrow{\bigcap_A} P(A) \xrightarrow{inf_A} A$$

is an internal functor and $sup_A \dashv \downarrow seg_A$.

Furthermore, $id_A = \{\ \}_A \circ sup_A$.

Proof. sup_A is an internal functor by construction.

i) $id_A = \uparrow seg_A \circ inf_A = \uparrow seg_A \circ \{\ \}_{P(A)} \circ \bigcap_A \circ inf_A =$

$\{\ \}_A \circ \exists_{\uparrow seg_A} \circ \bigcap_A \circ inf_A = \{\ \}_A \circ sup_A \leqslant \downarrow seg_A \circ sup_A =$

$\uparrow seg_A \circ \uparrow seg_{P(A)} \circ P(\uparrow seg_A) \circ \exists_{\uparrow seg_A} \circ \bigcap_A \circ inf_A \leqslant$

$\uparrow seg_A \circ \uparrow seg_{P(A)} \circ \bigcap_A \circ inf_A = \uparrow seg_A \circ inf_A = id_A$

i.e. $\{\ \}_A \circ sup_A = \downarrow seg_A \circ sup_A = id_A$.

ii) As $inf_A \circ \downarrow seg_A = \uparrow seg_{P(A)} \circ P(\uparrow seg_A) \perp \exists_{\uparrow seg_A} \circ \bigcap_A$ we have that

$sup_A \circ \downarrow seg_A = \exists_{\uparrow seg_A} \circ \bigcap_A \circ inf_A \circ \downarrow seg_A =$

$\exists_{\uparrow seg_A} \circ \bigcap_A \circ \uparrow seg_{P(A)} \circ P(\uparrow seg_A) \geqslant id_{P(A)}$

This concludes the proof of Proposition 2.5 □

Proposition 2.6. *A morphism $f_* : A \longrightarrow B$ between two internally complete ordered objects is inf-preserving iff it is an internal functor having a left adjoint f^*. If f_* is inf-preserving then*

$$f^* = B \xrightarrow{\uparrow seg_B} P(B) \xrightarrow{P(f_*)} P(A) \xrightarrow{inf_A} A$$

and f^* is sup-preserving.

Proof. If f_* is inf-preserving then f_* is an internal functor. As for f^*, as defined above, it is an internal functor independently of the properties of f_*.

i) $f^* \circ f_* = \uparrow seg_B \circ P(f_*) \circ inf_A \circ f_* = \uparrow seg_B \circ P(f_*) \circ \exists_{f_*} \circ inf_B \geqslant$

$\uparrow seg_B \circ inf_B = id_B$

ii) $f_* \circ f^* = f_* \circ \uparrow seg_B \circ P(f_*) \circ inf_A \leqslant \uparrow seg_A \circ inf_A = id_A$,

as f_* is an internal functor, whence $f^* \dashv f_*$.

Conversely, if f_* is an internal functor having a left adjoint $g : B \longrightarrow A$, then $g \circ \uparrow seg_A = \uparrow seg_B \circ P(f_*)$, and as

$\exists_{f_*} \circ inf_B \perp \uparrow seg_B \circ P(f_*)$ and

$inf_B \circ f_* \perp g \circ \uparrow seg_A$

it follows from the uniqueness theorem for internal adjoint functors that

$\exists_{f_*} \circ inf_B = inf_A \circ f_*$ and $g = f_*$.

Finally, if $f^* \dashv f_*$ then $\downarrow seg_A \circ P(f^*) = f_* \circ \downarrow seg_B$ and as

$\exists_{f^*} \circ sup_A \dashv \downarrow seg_A \circ P(f^*)$ and

$sup_B \circ f^* \dashv f_* \circ \downarrow seg_B$

it follows that $\exists_{f^*} \circ sup_A = sup_B \circ f^*$.

This concludes the proof of Proposition 2.6 □

Statement. The assignment

$(A, \uparrow seg_A, inf_A) \quad \longmapsto \quad (A, \downarrow seg_A, sup_A)$

$f_* : A \longrightarrow B \quad \longmapsto \quad f^* : B \longrightarrow A,$

defines a contravariant functor

$$I : \underline{C}(\underline{\mathbf{E}}) \longrightarrow \overline{C}(\underline{\mathbf{E}}).$$

Corollaries.

Corollary 2.7. *The internal power objects are internally co-complete.*

$\bigcup_X : PP(X) \longrightarrow P(X)$ *is given by*

$\bigcup_X = \exists_{\uparrow seg_{P(X)}} \circ \bigcap_{P(X)} \circ \bigcap_X .$

Corollary 2.8.

$\{\ \}_{P(X)} \circ \bigcup_X = id_{P(X)}.$

Corollary 2.9.

As $\exists_{\{\ \}_X} \circ \bigcup_X \dashv \downarrow seg_{P(X)} \circ P(\{\ \}_X) = id_{P(X)}$*, we have that*

$\exists_{\{\ \}_X} \circ \bigcup_X = id_{P(X)}.$

Corollary 2.10. *As* $\bigcup_X \dashv \downarrow seg_{P(X)}$*, we have that*

$\bigcup_{P(X)} \circ \bigcup_X = \exists_{\bigcup_X} \circ \bigcup_X .$

Corollary 2.11. $\forall f \in \mathrm{Hom}_{\underline{\mathbf{E}}}(X,Y)$ *we have* $\exists_f \dashv P(f)$*. It follows that*

$\exists_{\exists_f} \circ \bigcup_Y = \bigcup_X \circ \exists_f .$

Let $\exists\!\!| = (\exists, \{\ \}, \bigcup)$. In this notation $\exists\!\!|$ is a monad on $\underline{\mathbf{E}}$. $\exists\!\!|$ is called **the internal power monad on $\underline{\mathbf{E}}$**.

By dualizing the proofs of Proposition 2.5 and Proposition 2.6 we get

Proposition 2.5.* *Internally co-complete ordered objects are internally complete. Explicitly, if* $A = (\downarrow seg_A, sup_A)$ *is in* $\overline{C}(\underline{\mathbf{E}})$ *then*

$$inf_A = P(A) \xrightarrow{\exists_{\downarrow seg_A}} PP(A) \xrightarrow{\bigcap_A} P(A) \xrightarrow{sup_A} A$$

is a contravariant internal functor and $inf_A \perp \uparrow seg_A$*, furthermore* $id_A = \{\ \}_A \circ inf_A$.

Proposition 2.6.* *A morphism* $f^* : B \longrightarrow A$ *between two internally co-complete ordered objects is sup-preserving iff it is an internal functor having a right adjoint* f_**.If* f^* *is sup-preserving then*

$$f_* = A \xrightarrow{\downarrow seg_A} P(A) \xrightarrow{P(f^*)} P(B) \xrightarrow{sup_B} B$$

and f_* *is inf-preserving.*

Statement. The assignment

$$\begin{array}{lcl} (A, \downarrow seg_A, sup_A) & \longmapsto & (A, \uparrow seg_A, inf_A) \\ f^* : B \longrightarrow A & \longmapsto & f_* : A \longrightarrow B, \end{array}$$

defines a contravariant functor

$$J : \overline{C}(\underline{\mathbf{E}}) \longrightarrow \underline{C}(\underline{\mathbf{E}}).$$

By the uniqueness theorem for adjoint internal functors we get that I and J are antiisomorphisms of categories and that $J = I^{-1}$.

If $(A, \downarrow seg_A, sup_A) \in |\overline{C}(\underline{\mathbf{E}})|$ then the adjunction $sup_A \dashv \downarrow seg_A$ implies that

$$\bigcup\nolimits_A \circ sup_A = \exists_{sup_A} \circ sup_A \qquad \text{and} \qquad \{\ \}_A \circ sup_A = id_A.$$

It follows that the assignment

$$\overline{J}(A, \downarrow seg_A, sup_A) = (A, sup_A) \qquad \text{and} \qquad \overline{J}(f^*) = f^*$$

defines a covariant functor $\overline{J}$ from $\overline{C}(\underline{\mathbf{E}})$ to $\underline{\mathbf{E}}^{\exists|}$, the category of algebras for the internal power monad on $\underline{\mathbf{E}}$.

Proposition 2.7. *If* $(A, s) \in |\underline{\mathbf{E}}^{\exists|}|$ *then*

$$\downarrow seg(s) = A \xrightarrow{\{\ \}_A} P(A) \xrightarrow{P(s)} PP(A) \xrightarrow{\bigcup_A} P(A)$$

defines an internal order on A. Furthermore, s is an internal functor with respect to this ordering and $s \dashv \downarrow seg(s)$.

Proof. By assumption we have commutativity of

$$\begin{array}{ccc} A & \xrightarrow{\{\ \}_A} & P(A) \\ & {\scriptstyle id_A}\searrow & \downarrow{\scriptstyle s} \\ & & A \end{array} \qquad \begin{array}{ccc} PP(A) & \xrightarrow{\exists_s} & P(A) \\ \downarrow{\scriptstyle \bigcup_A} & & \downarrow{\scriptstyle s} \\ P(A) & \xrightarrow{s} & A \end{array}$$

1) $\downarrow seg(s) \circ s = \{\ \}_A \circ P(s) \circ \bigcup_A \circ s = \{\ \}_A \circ P(s) \circ \exists_s \circ\ s = \{\ \}_A \circ s = id_A.$

   In particular $\downarrow seg(s)$ is monic and $\exists_{\downarrow seg(s)} \leqslant P(s)$

2) $s \circ \downarrow seg(s) = s \circ \{\ \}_A \circ P(s) \circ \bigcup_A = \{\ \}_{P(A)} \circ \exists_s \circ P(s) \circ \bigcup_A \geqslant$

   $\{\ \}_{P(A)} \circ \bigcup_A = id_{P(A)}$

3) $\{\ \}_A = \{\ \}_A \circ \exists_{\{\ \}_A} \circ \bigcup_A \leqslant \{\ \}_A \circ P(s) \circ \bigcup_A = \downarrow seg(s),$

   as $\exists_{\{\ \}_A} \leqslant P(s)$ follows from $\{\ \}_A \circ s = id_A$.

4) $\downarrow seg_A \circ \exists_{\downarrow seg_s} \circ \bigcup_A = \downarrow seg(s) \circ P(s) \circ \bigcup_A \leqslant$

   $\downarrow seg(s) \circ P(s) \circ \bigcup_A \circ\ s \circ \downarrow seg(s) = \downarrow seg(s) \circ P(s) \circ \exists_s \circ\ \ s \circ \downarrow seg(s) =$

   $\downarrow seg(s) \circ\ \ s \circ \downarrow seg(s) = \downarrow seg(s)$. It follows that

4.1) $\downarrow seg(s) \circ P(s) \leqslant \downarrow seg(s) \circ \downarrow seg_{P(A)}$ and

4.2) $\downarrow seg(s) \leqslant \downarrow seg(s) \circ \downarrow seg_{P(A)} \circ P(\downarrow seg(s))$.

5) By 1), 3) and 4.2) we have that $\downarrow seg(s)$ is an internal ordering on A.

6) If $(B, \downarrow seg_B, sup_B) \in |\overline{C}(\underline{\mathbf{E}})|$ then $\downarrow seg_B = \{\ \}_B \circ P(sup_B) \circ \bigcup_B$.

Indeed, $\downarrow seg_B = \downarrow seg_B \circ \{\ \}_{P(B)} \circ \bigcup_B = \{\ \}_B \circ \exists_{\downarrow seg_B} \circ \bigcup_B \leqslant$

$\{\ \}_B \circ P(sup_B) \circ \bigcup_B \leqslant \downarrow seg_B \circ P(sup_B) \circ \bigcup_B = \downarrow seg_B$.

Notice that $\exists_{\downarrow seg_B} \leqslant P(sup_B)$ follows from $\downarrow seg_B \circ sup_B = id_B$. The last equality follows from $sup_B \dashv \downarrow seg_B$ via Proposition 2.6*. In particular we have

$$\downarrow seg_{P(A)} = \{\ \}_{P(A)} \circ P(\textstyle\bigcup_A) \circ \bigcup_{P(A)}.$$

Finally, as $\{\ \}_A \circ P(s) \circ P(\exists_s) \circ \bigcup_{P(A)} \circ \exists_s =$

$\{\ \}_A \circ P(s) \circ P(\exists_s) \circ \exists_{\exists_s} \circ \bigcup_A \leqslant \{\ \}_A \circ P(s) \circ \bigcup_A = \downarrow seg(s)$

it follows that

$\downarrow seg(s) \circ \downarrow seg_{P(A)} = \downarrow seg(s) \circ \{\ \}_{P(A)} \circ P(\bigcup_A) \circ \bigcup_{P(A)} =$

$\{\ \}_A \circ \exists_{\downarrow seg(s)} \circ P(\bigcup_A) \circ \bigcup_{P(A)} \leqslant \{\ \}_A \circ P(s) \circ P(\bigcup_A) \circ \bigcup_{P(A)} =$

$\{\ \}_A \circ P(\bigcup_A \circ\ s) \circ \bigcup_{P(A)} = \{\ \}_A \circ P(\exists_s \circ\ s) \circ \bigcup_{P(A)} =$

$\{\ \}_A \circ P(s) \circ P(\exists_s) \circ \bigcup_{P(A)} \leqslant \downarrow seg(s) \circ P(s)$.

7) From 4.1) and 6) we have that $\downarrow seg(s) \circ P(s) = \downarrow seg(s) \circ \downarrow seg_{P(A)}$.

It follows that s is an internal functor and that s is the left adjoint of $\downarrow seg(s)$.

This concludes the proof of Proposition 2.7. □

From Proposition 2.7 we get that the assignment

$$\overline{I}(A, s) = (A, \downarrow seg(s), s) \qquad \text{and} \qquad \overline{I}(f) = f$$

defines a covariant functor $\overline{I}$ from $\underline{\mathbf{E}}^{\exists|}$ to $\overline{C}(\underline{\mathbf{E}})$, and that the composite $\overline{J} \circ \overline{I} = id_{\underline{\mathbf{E}}^{\exists|}}$. From the proof, 6), of the proposition it follows that $\overline{I} \circ \overline{J} = id_{\overline{C}(\underline{\mathbf{E}})}$.

As a consequence we have for an elementary topos $\underline{\mathbf{E}}$ the theorem which in the case $\underline{\mathbf{E}} = \underline{Sets}$ was first established by E. Manes [18].

Theorem 2.1. *The category $\overline{C}(\underline{\mathbf{E}})$ of internally co-complete ordered objects and sup-preserving morphisms is tripleable over $\underline{\mathbf{E}}$ and isomorphic to the Eilenberg-Moore category $\underline{\mathbf{E}}^{\exists|}$ for the internal power monad on $\underline{\mathbf{E}}$. Also, it is antiisomorphic to $\underline{C}(\underline{\mathbf{E}})$, the category of internally complete ordered objects and inf-preserving morphisms.*

The standard argument that $\underline{\mathbf{E}}^{\exists|}$ has finite inverse limits can also be applied to the forgetful functor

$$\underline{C}(\underline{\mathbf{E}}) \longrightarrow \underline{\mathbf{E}}$$

and consequently we have that $\overline{C}(\underline{\mathbf{E}})$ has finite inverse limits and finite direct limits as well.

Given a finite diagram $\underline{D}$ in $\overline{C}(\underline{\mathbf{E}})$, pass to $\underline{C}(\underline{\mathbf{E}})$ and compute its inverse limit (as in $\underline{\mathbf{E}}$) and return to $\overline{C}(\underline{\mathbf{E}})$. The result is the direct limit in $\overline{C}(\underline{\mathbf{E}})$ of $\underline{D}$.

In particular $\overline{C}(\underline{\mathbf{E}})$ has finite bi-products (the cartesian product) and a zero object (1).

Notice that the diagonal as well as the terminal morphism on any object in $\overline{C}(\underline{\mathbf{E}})$ are both inf- and sup-preserving. It follows that any $(A, \downarrow seg_A, sup_A) \in |\overline{C}(\underline{\mathbf{E}})|$ is a bounded lattice object in $\underline{\mathbf{E}}$. In particular we rediscover the already established fact that the internal power objects are upper semilattices in $\underline{\mathbf{E}}$.

Due to Theorem 2.1 the objects $\overline{C}(\underline{\mathbf{E}})$, $\underline{C}(\underline{\mathbf{E}})$ and $\underline{\mathbf{E}}^{\exists|}$ will simply be called complete lattices (in $\underline{\mathbf{E}}$).

Theorem 2.2. *Let A be a complete lattice in $\underline{\mathbf{E}}$ and let $f : A \longrightarrow A$ be an internal endofunctor on A, then f has a smallest fixpoint (defined over 1).*

Proof. Consider the following diagram

$$\begin{array}{ccccccc}
P(A) & \xrightarrow{inf_A} & A & \xrightarrow{<f, id_A>} & A \times A & \xrightarrow{\uparrow \widehat{seg}_A} & \Omega \\
\uparrow \exists_i & & \uparrow i & & P.B. & & \uparrow true \\
P(F) & & F & & \longrightarrow & & 1
\end{array}$$

If $a : N \longrightarrow A$ is any element in A then a factors through F iff $a \circ f \leqslant a$. In particular $i \circ f \leqslant i$. But $i \circ f \leqslant i$ iff $\uparrow seg_A \circ P(i \circ f) \leqslant \uparrow seg_A \circ P(i)$ iff $\exists_{i \circ f} \circ inf_A \leqslant \exists_i \circ inf_A$, and as $inf_A \circ f \leqslant \exists_i \circ inf_A$ as f is an internal functor, it follows that

$$\exists_i \circ inf_A \circ f \leqslant \exists_i \circ \exists_f \circ inf_A = \exists_{i \circ f} \circ inf_A \leqslant \exists_i \circ inf_A,$$

and so there exists a uniquely determined morphism

$$inf_F : P(F) \longrightarrow F$$

such that

$$\exists_i \circ inf_A = inf_F \circ i.$$

It follows that $(F, \uparrow seg_F = i \circ \uparrow seg_A \circ P(i), inf_F) \in |\underline{C}(\underline{\mathbf{E}})|$, and that i is a $\underline{C}(\underline{\mathbf{E}})$-morphism. In particular

$$a = \ulcorner true_F \urcorner \circ \exists_i \circ inf_A = \ulcorner true_F \urcorner \circ inf_F \circ i$$

is the smallest global section in F.

Now $a \circ f \leqslant a$, and as f is an internal functor $a \circ f \circ f \leqslant a \circ f$. Thus $a \circ f$ factors through F and so $a \leqslant a \circ f$. It follows that $a \circ f = a$.

If $b : N \longrightarrow A$ is any fixpoint for f, $b \circ f = b$, then $b \circ f \leqslant b$ and so b factors through F. Again $!_N \circ a \leqslant b$ by the minimality of a.

This concludes the proof of Theorem 2.2. □

Remark. In the category of sets Theorem 2.2 is known as Tarski's Fixpoint Theorem, [25]. In the case $A = P(X)$ it was first published by Knaster [11] as the result of joint work of Knaster and Tarski, but it is easy to trace the theorem through Zermelo to Dedekind [2]. E.g. the following proposition goes back to **Was sind und was sollen die Zahlen**. The number 5.41 refers to **Aspects of Topoi** where the proposition was established by Freyd by another method.

Proposition 5.41. *Given $x : 1 \longrightarrow X$ and $t : X \longrightarrow X$ in an elementary topos $\underline{\mathbf{E}}$ then there exists a subobject*

$$i : Y \rightarrowtail X$$

such that $im(i \circ t) \vee x = Y$.

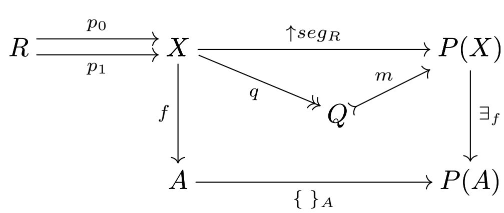
Proof. Apply the fixpoint theorem in the case $A = P(X)$ and $f = \langle \exists_t, !_{P(X)} \circ x \circ \{\ \}_X \rangle \circ \vee_{P(X)}$. □

Actually, we get a smallest solution of the problem.

The original reason for establishing the fixpoint theorem for elementary topoi was to verify the following

Proposition 2.8. *Elementary topoi have coequalizers.*

Proof. First we notice that equivalence relations have coequalizers.



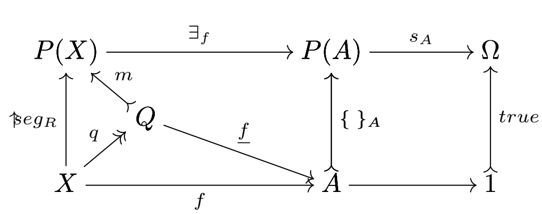
Let $\langle p_0, p_1 \rangle : R \rightarrowtail X \times X$ be an equivalence relation in $\underline{\mathbf{E}}$ and let $\uparrow seg_R$ be the exponential adjoint of $ch_{X\times X}(R)$. As R is an equivalence relation we conclude that $\forall M \in |\underline{\mathbf{E}}| \quad \forall x, y \in X :$ $x \circ \uparrow seg_R = y \circ \uparrow seg_R$ iff xRy In particular we have that : $p_0 \circ \uparrow seg_R = p_1 \circ \uparrow seg_R$.

Let $q \circ m = \uparrow seg_R$ be the epi-mono-factorization of $\uparrow seg_R$ (which exists in $\underline{\mathbf{E}}$ by Corollary 2.4), then $p_0 \circ q = p_1 \circ q$ and $\langle p_0, p_1 \rangle$ is the kernel-pair of q.

We claim that q is the coequalizer of p_0 and p_1.

Let $f \in \mathrm{Hom}_{\underline{\mathbf{E}}}(X, A)$ be a morphism in $\underline{\mathbf{E}}$ such that $p_0 \circ f = p_1 \circ f$. Then $\uparrow seg_R \circ \exists_f = f \circ \{\ \}_A$. Indeed, $\{\ \}_A \leqslant \uparrow seg_R$ as R is reflexive, and so $f \circ \{\ \}_A = \{\ \}_X \circ \exists_f \leqslant \uparrow seg_R \circ \exists_f$.

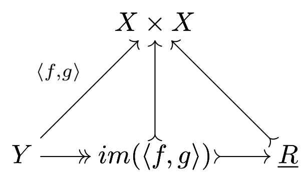
Now $\uparrow seg_R \circ \exists_f \leqslant f \circ \{\ \}_A$ iff $\uparrow seg_R \leqslant f \circ \{\ \}_A \circ P(f)$. But the latter inequality is the assumption on f. It follows that $\uparrow seg_R \circ \exists_f = f \circ \{\ \}_A$.



Now $q \circ m \circ \exists_f \circ s_A = \uparrow seg_R \circ \exists_f \circ s_A = f \circ \{\ \}_A \circ s_A = true_X = q \circ true_Q$, it follows that $m \circ \exists_f \circ s_A = true_Q$ as q is epic. Thus there exists a factorization $\underline{f} : Q \longrightarrow A$ such that $\underline{f} \circ \{\ \}_A = m \circ \exists_f$, and as $\{\ \}_A$ is monic it follows that $q \circ \underline{f} = f$. As q is epic we see that $q = eq(p_0, p_1)$.

This proves that equivalence relations have coequalizers.

As for the general case let $f, g \in \mathrm{Hom}_{\underline{\mathbf{E}}}(Y, X)$. Now, provided we can construct the smallest equivalence relation $\underline{R}$ on X which contains the image of $\langle f, g\rangle$, it follows from the above and from the fact any kernel-pair is an equivalence relation that the coequalizer of f and g exists and equals the coequalizer of $\underline{R}$.



Consider the following internal functors on $P(X \times X)$:

f_1 = "Adjoining the image of $\langle f, g\rangle$"

$= \langle id_{P(X\times X)}, !_{P(X\times X)} \circ \ulcorner ch(im(\langle f, g\rangle))\urcorner\rangle \circ \vee_{P(X\times X)}$

f_2 = "Adjoining the diagonal on X"

$= \langle id_{P(X\times X)}, !_{P(X\times X)} \circ \ulcorner \delta_X \urcorner\rangle \circ \vee_{P(X\times X)}$

f_3 = "Taking the inverse of the relation" $= \exists_{tw_{X,X}}$

f_4 = "Taking the square of a relation"

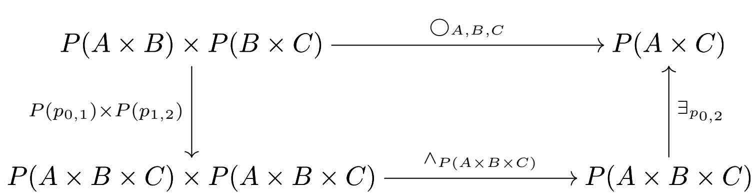
$= \Delta_{P(X\times X)} \circ \bigcirc_{X,X,X}$ ✠

and let $g_1 = f_1$, $g_{n+1} = \langle g_n, f_{n+1}\rangle \circ \vee_{P(X\times X)}$ for $n = 1, 2, 3$, then the fixpoints of g_4 (over 1) are exactly the equivalence relations on X containing the image of $\langle f, g\rangle$, and by the fixpoint theorem there exists a smallest such fixpoint.

This concludes the proof of Proposition 2.8 □

✠

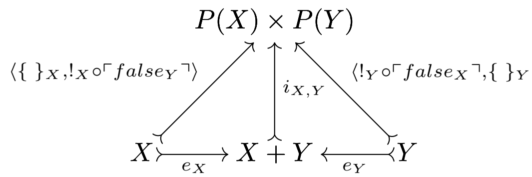
Using the epi-mono-factorization we have composition of relations in $\underline{\mathbf{E}}$. This construction is internalized as follows:



Observe that from the proof of Proposition 2.8 it follows that epimorphisms are coequalizers (of their kernel-pairs).

Proposition 2.9. *Elementary topoi have finite coproducts.*

Proof. Let $X, Y \in |\underline{\mathbf{E}}|$ and let $i_{X,Y} : X + Y \rightarrowtail P(X) \times P(Y)$ be the smallest subobject of $P(X) \times P(Y)$ containing the two subobjects $\langle \{\ \}_X, !_X \circ \ulcorner false_Y \urcorner \rangle$ and $\langle !_Y \circ \ulcorner false_X \urcorner, \{\ \}_Y \rangle$

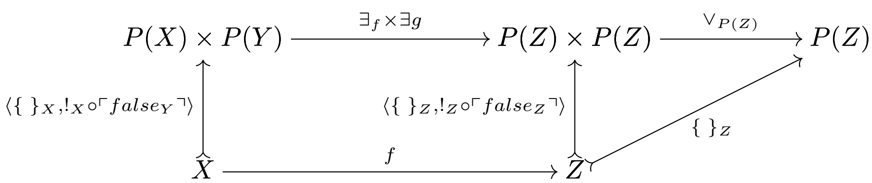


If $ch(X)$ and $ch(Y)$ denote the characters of these subobjects then

$$ch_{P(X)\times P(X)}(i_{X,Y}) = \langle ch(X), ch(Y)\rangle \circ \vee.$$

It follows from the defining property of $i_{X,Y}$ that the two morphisms e_X and e_Y are joint epi, as $\underline{\mathbf{E}}$ has equalizers.

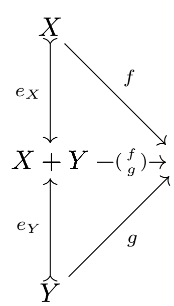
Let $f \in \mathrm{Hom}_{\underline{\mathbf{E}}}(X, Z)$ and $g \in \mathrm{Hom}_{\underline{\mathbf{E}}}(Y, Z)$ and consider the following diagram.



Now $\ulcorner false_Z \urcorner$ is the smallest global section in $P(Z)$ and therefore the triangle is commutative. The square is commutative as $\exists_g \dashv P(g)$ and as $\{\ \}$ is natural. It follows that the inverse image of $\{\ \}_Z$ along $\exists_f \times \exists g \circ \vee_{P(Z)}$ contains $\langle \{\ \}_X, !_X \circ \ulcorner false_Y \urcorner \rangle$. Dually, it also contains the subobject $\langle !_Y \circ \ulcorner false_X \urcorner, \{\ \}_Y \rangle$, and therefore it contains the smallest subobject with this property. I.e. there exists a morphism

$$\binom{f}{g} : X + Y \longrightarrow Z$$

such that $i_{X,Y} \circ (\exists_f \times \exists_g) \circ \vee_{P(X)} = \binom{f}{g} \circ \{\ \}_Z$. It follows that



is commutative. This means that

$$X \underset{e_X}{\rightarrowtail} X + Y \underset{e_Y}{\leftarrowtail} Y$$

is a coproduct in $\underline{\mathbf{E}}$ as e_X and e_Y are joint epi.

This concludes the proof of Proposition 2.9. $\square$

Remark.It follows from the above construction that coproducts in an elementary topos are disjoint.

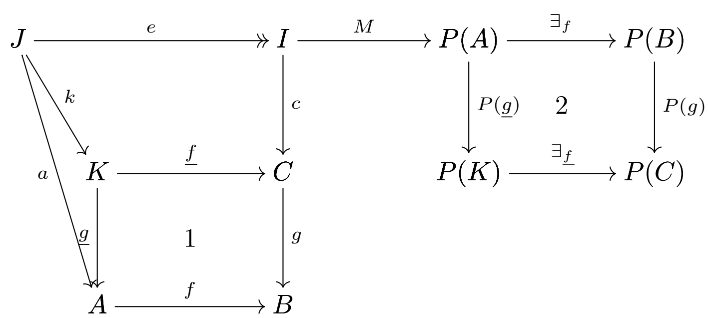
Theorem 2.3. *Elementary topoi have finite colimits.*

Proof. This is the content of Corollary 2.1, Proposition 2.8 and Proposition 2.9. □

Remark. Due to Theorem 1.1 and Theorem 2.3 we now know that the definition of an elementary topos given in this work agrees with the original one [5]. In cases where we want to establish that a certain category is an elementary topos Theorem 2.3 will reduce the amount of work required for performing the proof. A typical example is given in [9].

One of the most frequently quoted properties of elementary topoi is that they satisfies the Beck condition for pull backs. We shall use this property several times in the following internal form.

Consider the following diagram



If $\underline{g} \circ f = \underline{f} \circ g$ then $P(f) \circ P(\underline{g}) = P(g) \circ P(\underline{f})$ and therefore $P(\underline{g}) \circ \exists_{\underline{f}} \leqslant \exists_f \circ P(g)$. In this notation we have the following

Proposition 2.10. *If the diagram 1 is a pull back then the diagram 2 is commutative.*

Proof. Let $I \in \underline{\mathbf{E}}$, $M \in P(A)$ and $c \in C$ such that $c \mathrel{\underline{\in}} M \circ \exists_f \circ P(g)$. We claim that $c \mathrel{\underline{\in}} M \circ P(\underline{g}) \circ \exists_{\underline{f}}$.

By the assumption we have that $c \circ g \mathrel{\underline{\in}} M \circ \exists_f$. Thus $\exists J \in |\underline{\mathbf{E}}|$, $\exists e \in I$ (epi), $\exists a \in A$ such that $a \mathrel{\underline{\in}} e \circ M$ and $e \circ c \circ g = a \circ f$. As 1 is a pull back there exists $k \in \mathrm{Hom}_{\underline{\mathbf{E}}}(J, K)$ such that $k \circ \underline{g} = a$ and $k \circ \underline{f} = e \circ c$, and as $k \circ \underline{g} = a \mathrel{\underline{\in}} e \circ M$ and e is epic, this proves that $c \mathrel{\underline{\in}} M \circ P(\underline{g}) \circ \exists_{\underline{f}}$. □

Remarks.

1). Notice that the above proof did not depend on the uniqueness of k.

2). From $P(\underline{g}) \circ \exists_{\underline{f}} = \exists_f \circ P(g)$ we derive $P(\underline{f}) \circ \forall_{\underline{g}} = \forall_g \circ P(f)$ by uniqueness of adjoints of internal functors.

3). In any regular category pulling back along an epimorphism reflects monics. In the above notation this means that if g is epic and $\underline{f}$ is monic and 1 is a pull back then f is a monomorphism.

As elementary topoi are regular categories, it follows that they have this property and the proof from the regular case applies. We have, however, the following concise proof.

As $P(\underline{g}) \circ \exists_{\underline{f}} = \exists_f \circ P(g)$ and as $\underline{g}$ is epic and $\underline{f}$ is monic we have that $P(\underline{g})$ and $\exists_{\underline{f}}$ are monic. Thus $\exists_f$ is monic, and therefore f is also a monomorphism.

Most categorical properties of elementary topoi reflect themselves internally. For example, the following proposition is the internal version of the fact that the cartesian product of two epimorphisms is an epimorphism (cf. Theorem 1.1).

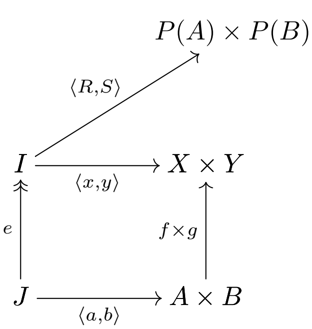
Proposition 2.11. *$\tilde{p} = \{\tilde{p}_{A,B} : P(A) \times P(B) \longrightarrow P(A \times B)\}_{(A,B)\in|\underline{\mathbf{E}}|\times|\underline{\mathbf{E}}|}$ is a natural transformation with respect to $\exists$.*

Proof. $\tilde{p}_{A,B}$ was defined in Chapter 1, (8). The statement of the proposition means that all diagrams of the form

$$\begin{array}{ccc} P(X)\times P(Y) & \xrightarrow{\tilde{p}_{X,Y}} & P(X\times Y) \\ \uparrow{\scriptstyle \exists_f\times\exists_g} & & \uparrow{\scriptstyle \exists_{f\times g}} \\ P(A)\times P(B) & \xrightarrow{\tilde{p}_{A,B}} & P(A\times B) \end{array}$$

are commutative.

We shall give a detailed proof of this fact by means of the elementary descriptions involved, and it may serve as a prototype of this kind of proof.

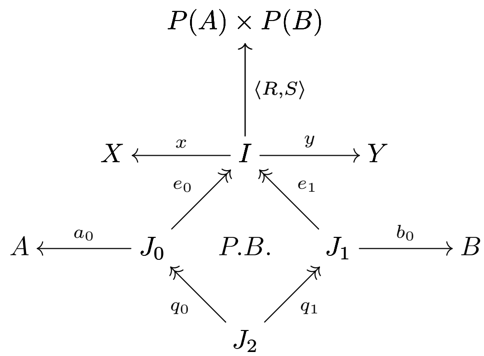


$\forall I \in |\underline{\mathbf{E}}| \qquad \forall R \in P(A) \qquad \forall S \in P(B) \qquad \forall x \in X \qquad \forall y \in Y :$

$\langle x, y\rangle \underline{\in} \langle R, S\rangle \circ\ \tilde{p}_{A,B} \circ \exists_{f\times g}$ iff

$\exists J \in |\underline{\mathbf{E}}| \qquad \exists e \in I\ \text{(epi)} \qquad \exists a \in A \qquad \exists b \in B$ such that

$a \underline{\in} e \circ R$, $b \underline{\in} e \circ S$, $a \circ f = e \circ x$ and $b \circ g = e \circ y$.



$\forall I \in |\underline{\mathbf{E}}| \qquad \forall R \in P(A) \qquad \forall S \in P(B) \qquad \forall x \in X \qquad \forall y \in Y :$

$\langle x, y\rangle \mathrel{\underline{\in}} \langle R, S\rangle \circ \; \exists_f \times \exists_g \circ \tilde{p}_{X,Y}$ iff

$\exists J_0 \in |\underline{\mathbf{E}}| \qquad \exists e_0 \in I$ (epi) $\qquad \exists a_0 \in A$ such that

$a_0 \mathrel{\underline{\in}} e_0 \circ R$ and $a_0 \circ f = e_0 \circ x$ and

$\exists J_1 \in |\underline{\mathbf{E}}| \qquad \exists e_1 \in I$ (epi) $\qquad \exists b_0 \in B$ such that

$b_0 \mathrel{\underline{\in}} e_1 \circ S$ and $b_0 \circ g = e_1 \circ y$.

By taking $J_0 = J_1 = J$, $e_0 = e_1 = e$, $a_0 = a$ and $b_0 = b$, it follows that

$$\tilde{p}_{A,B} \circ \exists_{f\times g} \leqslant \exists_f \times \exists_g \circ \tilde{p}_{X,Y}$$

We can prove the other inequality as follows. By Proposition 2.4 the two morphisms q_0 and q_1 in the indicated pull back are epics. Thus by taking $J = J_2$, $e = q_0 \circ e_0 = (q_1 \circ e_1)$, $a = q_0 \circ a_0$ and $b = q_1 \circ b_1$, it follows that

$$\tilde{p}_{A,B} \circ \exists_{f\times g} \geqslant \exists_f \times \exists_g \circ \tilde{p}_{X,Y}$$

This concludes the proof of Proposition 2.11. □

Recall the notion of internal bi-functors that admit an exponential. (The concept was described in Chapter 1, (9)

Lemma 2.1. *Let $A,\ B,\ C,\ A_0,\ B_0$ and C_0 be internally ordered objects, let*

$$\diamond : A \times B \longrightarrow C$$

be an internal bi-functor which admit an exponential

$$\rightarrowtriangle : B^{op} \times C \longrightarrow A,$$

and let $g : B_0 \longrightarrow B$ be an internal functor and

$$A_0 \underset{f_*}{\overset{f^*}{\rightleftarrows}} A \qquad \text{and} \qquad C \underset{h_*}{\overset{h^*}{\rightleftarrows}} C_0$$

be a pair of adjoint internal functors, $f^* \dashv f_*$ and $h^* \dashv h_*$, then

$$A_0 \times B_0 \xrightarrow{f^* \times g} A \times B \xrightarrow{\diamond} C \xrightarrow{h^*} C_0 \qquad \textit{expo}$$

$$B_0 \times C_0 \xrightarrow{g \times h_*} B \times C \xrightarrow{\rightarrow} A \xrightarrow{f_*} A_0$$

Proof. $\forall N \in |\underline{\mathbf{E}}| \quad \forall a \in A_0 \quad \forall b \in B_0 \quad \forall c \in C_0 :$

$(a \circ f^* \diamond b \circ g) \circ h^* \leqslant c \quad$ iff $\quad a \circ f^* \diamond b \circ g \leqslant c \circ h_* \quad$ iff

$a \circ f^* \leqslant \ b \circ g \to c \circ h_* \quad$ iff $a \leqslant (\ b \circ g \to c \circ h_*) \circ f_*$

□

Let

$$\bullet : X \times Y \longrightarrow Z$$

be any (binary) morphism in $\underline{\mathbf{E}}$, and let

$$\underline{\bullet} : P(X) \times P(Y) \longrightarrow P(Z)$$

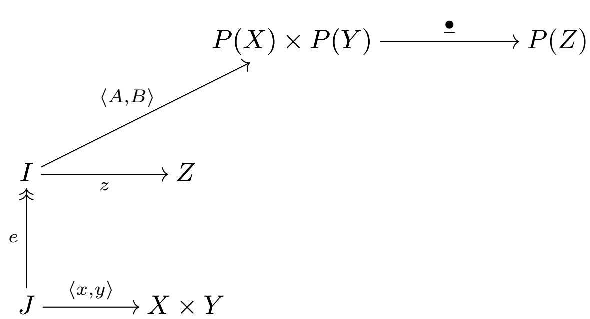
be the morphism

$$\underline{\bullet} = \tilde{p}_{X,Y} \circ \exists_{\bullet} = P(p_0) \times P(p_1) \circ \wedge_{P(X \times Y)} \circ \exists_{\bullet}.$$

By Lemma 2.1 the induces multiplication $\underline{\bullet}$ admits an exponential

$$\overrightarrow{\bullet} = P(p_1) \times P(\bullet) \circ \Rightarrow_{P(X \times Y)} \circ \forall_{p_0}$$

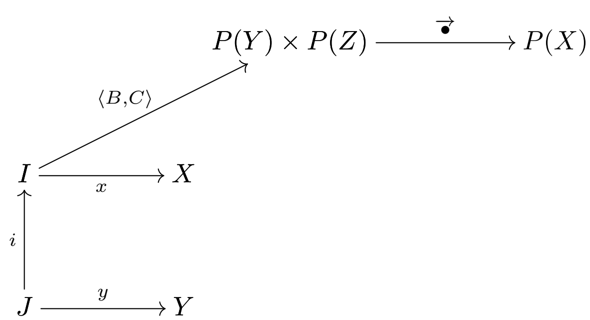
By the above construction we have the following elementary description of $\underline{\bullet}$ and $\overrightarrow{\bullet}$:



$\forall I \in |\underline{\mathbf{E}}| \quad \forall A \in P(X) \quad \forall B \in P(Y) \quad \forall z \in Z : z \ \underline{\in}\ A \underline{\bullet} B \quad$ iff

$\exists J \in |\underline{\mathbf{E}}| \quad \exists e \in I$ (epi) $\quad \exists x \in X \quad \exists y \in Y \quad$ such that

$x \ \underline{\in}\ e \circ A$, $y \ \underline{\in}\ e \circ B$ and $x \bullet y = e \circ z$.



$$\forall I \in |\underline{\mathbf{E}}| \qquad \forall B \in P(Y) \qquad \forall C \in P(Z) \qquad \forall x \in X : x \underline{\in} B \overrightarrow{\bullet} C \quad \text{iff}$$

$$\forall J \in |\underline{\mathbf{E}}| \qquad \forall y \in Y : y \underline{\in} i \circ B \quad \text{implies} \quad (i \circ x) \bullet y \underline{\in} i \circ C.$$

Proposition 2.12. *Let A, B and C be complete lattices in $\underline{\mathbf{E}}$ and let $\bullet : A \times B \longrightarrow C$ be an internal bi-functor. Then $\bullet$ admits an exponential $\rightarrow$ iff the following diagram commutes*

$$\begin{array}{ccccc} P(A) \times B & \xrightarrow{id_{P(A)} \times \{\ \}_B} & P(A) \times P(B) & \xrightarrow{\underline{\bullet}} & P(C) \\ \downarrow{\scriptstyle sup_A \times id_B} & & & & \downarrow{\scriptstyle sup_C} \\ A \times B & & \xrightarrow{\bullet} & & C \end{array}$$

Furthermore, if the square is commutative, then $\rightarrow$ is given by

$$\rightarrow\ = B \times C \xrightarrow{\{\ \}_B \times \downarrow seg_C} P(B) \times P(C) \xrightarrow{\overrightarrow{\bullet}} P(A) \xrightarrow{sup_A} A.$$

The proof of Proposition 2.12 is left to the reader. It is - using the elementary description of $\underline{\bullet}$ and $\overrightarrow{\bullet}$ - a direct translation from the classical proof in $\underline{Sets}$.

Corollary 2.12. *Let A be a complete lattice in $\underline{\mathbf{E}}$ and let $\bullet : A \times A \longrightarrow A$ be an internal symmetric bi-functor, then $\bullet$ admits an exponential $\rightarrow$ iff the following diagram commutes.*

$$\begin{array}{ccc} P(A) \times P(A) & \xrightarrow{\underline{\bullet}} & P(A) \\ \downarrow{\scriptstyle sup_A \times sup_A} & & \downarrow{\scriptstyle sup_A} \\ A \times A & \xrightarrow{\bullet} & A \end{array}$$

Observe that in the case $\bullet = \wedge_A$ then the Corollary 2.12 is the well known criterion for A to be a complete Heyting algebra (object in $\underline{\mathbf{E}}$).

In the below theorem let $p^0 = \ulcorner true \urcorner = \{\ \}_1 : 1 \longrightarrow P(1)$.

Theorem 2.4. $\mathbb{E} = ((\exists, \tilde{p}, p^0), \{\ \}, \bigcup)$ *is a symmetric monoidal monad on* $\underline{\mathbf{E}}$

Proof. We have seen that $\tilde{p}$ is a natural transformation. Notice that all cases of $\tilde{p}$ have a left adjoint k

$$k_{A,B} : P(A \times B) \xrightarrow{\langle \exists_{p_0}, \exists_{p_1} \rangle} P(A) \times P(B),$$

$k_{A,B} \dashv \tilde{p}_{A,B}$. Applying k we easily get that $(\exists, \tilde{p}, p^0)$ is a monoidal functor.

To say that $\{\ \} : id_{\underline{\mathbf{E}}} \Rightarrow \exists$ is a monoidal transformation means that

$$\begin{array}{ccc} 1 & \longrightarrow & 1 \\ & {\scriptstyle p^0} \searrow & \downarrow{\scriptstyle \{\ \}_1} \\ & & P(1) \end{array}$$

is commutative and that all cases of

$$\begin{array}{ccc} A\times B & & \\ \Big\downarrow{\scriptstyle \{\,\}_A\times\{\,\}_B} & \searrow^{\{\,\}_{A\times B}} & \\ P(A)\times P(B) & \xrightarrow[\tilde{p}_{A,B}]{} & P(A\times B) \end{array}$$

are commutative, - which clearly is the case.

To say that $\bigcup : \exists\circ\exists \Rightarrow \exists$ is a monoidal transformation means that

$$\begin{array}{ccccc} 1 & \xrightarrow{p^0} & P(1) & \xrightarrow{\exists_{p^0}} & PP(1) \\ & \searrow_{p^0} & & & \Big\downarrow{\scriptstyle \bigcup_1} \\ & & & & P(1) \end{array}$$

commutes, and that all cases of the following diagram are commutative.

$$\begin{array}{ccccc} PP(A)\times PP(B) & \xrightarrow{\tilde{p}_{P(A),P(B)}} & P(P(A)\times P(B)) & \xrightarrow{\exists_{\tilde{p}_{A,B}}} & PP(A\times B) \\ \Big\downarrow{\scriptstyle \bigcup_A\times\bigcup_B} & & & & \Big\downarrow{\scriptstyle \bigcup_{A\times B}} \\ P(A)\times P(B) & & \xrightarrow[\tilde{p}_{A,B}]{} & & P(A\times B) \end{array}$$

The triangle is commutative as $\exists_{\{\,\}_1}\circ\bigcup_1 = id_{P(1)}$ and the square s commutative by the Corollary 2.12 as the internal bi-functor $\tilde{p}_{A,B}$ admits an exponential.

Finally, $\tilde{p}$ is symmetric as the binary internal intersection on internal power objects is symmetric.

This concludes the proof of Theorem 2.4 □

From the general theory of monoidal functors (and monads), [7], we get that ∃ has a cotensorial strength

$$\lambda_{X,Y} : P(X^Y) \longrightarrow P(Y)^X$$

which is given by its exponential adjoint, namely

$$id_{P(X^Y)}\times\{\,\}_X\circ\tilde{p}_{Y^X,X}\circ\exists_{ev_{X,Y}} : P(Y^X)\times X \longrightarrow P(Y).$$

Combining this with the fact that the functors $(\ )^X$ on $\underline{\mathbf{E}}$ are left exact yields the proof of the following

Proposition 2.13. *The functor $(\ )^X$ preserves complete lattices in $\underline{\mathbf{E}}$. Explicitly, if A is a complete lattice in $\underline{\mathbf{E}}$, then A^X is a complete lattice under the pointwise ordering, i.e.*

$$P(A^X)\xrightarrow{\lambda_{X,A}} P(A)^X \xrightarrow{(sup_A)^X} A^X$$

is the internal sup on A^X. Dually for inf. Furthermore, if $f\in \mathrm{Hom}_{\underline{\mathbf{E}}}(X,Y)$ then A^f is sup- and inf-preserving and preserves all finitary operations existing on A.

We shall now establish a number of properties of internal functors on Heyting algebras in $\underline{\mathbf{E}}$. Most of then have most certainly occurred in articles of mathematics but at least one of them, explaining why f^{-1} of an open continuous function on topological spaces must preserve the formation of implication of open sets, seems to be new.

Proposition 2.14 (The Frobenius Reciprocity, [14])**.** *Let*

$$A \underset{g}{\overset{f}{\rightleftarrows}} B\,, \qquad f \dashv g$$

be a pair of internal adjoint functors on Heyting algebras in $\underline{\mathbf{E}}$*, then the following statements are equivalent.*

1) f preserves ↓-segments.

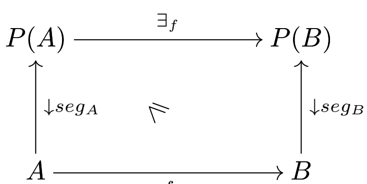
2) (f, g) satisfies the Frobenius Reciprocity Law.

3) g preserves implication.

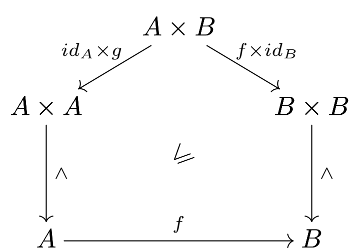
In case the internal functor g has a right adjoint h, each of these conditions is equivalent to

4) (f, g, h) is a Stone morphism.

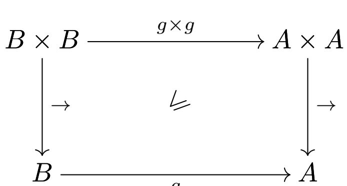
Proof. Consider the following diagrams.



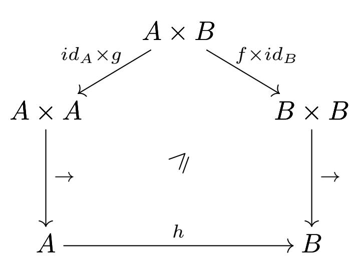
(1)



(2)



(3)



(4)

The indicated inequalities are valid and may easily be derived from the general rules of adjointness of internal functors. The exact meaning of condition n) is that the diagram (n) commutes.

1) $\Rightarrow$ 2). Let $I \in |\underline{\mathbf{E}}|$, $a \in A$ and $b \in B$, then $(a \wedge b \circ g) \circ f \leqslant a \circ f \wedge b \leqslant a \circ f$. From the last inequality we have that

$$a \circ f \wedge b \mathrel{\underline{\in}} a \circ f \circ \downarrow seg_B = a \circ \downarrow seg_A \circ \exists_f,$$

and therefore $\exists\, J \in |\underline{\mathbf{E}}|\ \ \exists\, e \in I(\text{epi})\ \exists\, x \in A$ such that $x \leqslant e \circ a$ and $x \circ f = e \circ a \circ f \wedge e \circ b$.

As $x \circ f \leqslant e \circ b$ we have $x \leqslant e \circ b \circ g$ and so $x \circ f = (e \circ a \wedge x) \circ f \leqslant (e \circ a \wedge e \circ b \circ g) \circ f \leqslant e \circ a \circ f \wedge e \circ b = x \circ f$, it follows that we have $(e \circ a \wedge e \circ b \circ g) \circ f = e \circ a \circ f \wedge e \circ b$, and as e is epic we finally get that $(a \wedge b \circ g) \circ f = a \circ f \wedge b$.

2) $\Rightarrow$ 1).Let $I \in |\underline{\mathbf{E}}|$, $a \in A$ and $b \in B$ such that $b \mathrel{\underline{\in}} a \circ f \circ \downarrow seg_B$, i.e. such that $b \leqslant a \circ f$. By 2) we get that $b = a \circ f \wedge b = (a \wedge b \circ g) \circ f$ and therefore $b \mathrel{\underline{\in}} a \circ \downarrow seg_A \circ \exists_f$ as $a \wedge b \circ g \leqslant a$.

2 $\iff$ 3) $\iff$ 4) is a corollary of Lemma 2.1. □

Corollary 2.13. *Let (A, B, f, g) be as in Proposition 2.14. If f is monic and g preserves implication, then f preserves binary intersection.*

Proof. As f is monic we have that $id_A = f \circ g$, and the statement follows from sticking $id_A \times f$ on top of the commutative pentagon (2). □

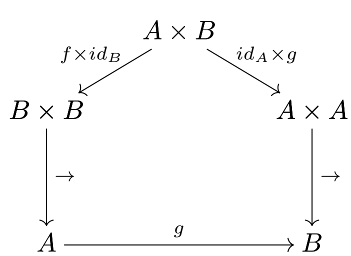
Proposition 2.15. *Let (A, B, f, g) be as in Proposition 2.14, then f is epic iff $g \circ f = id_B$ iff f preserves $\uparrow$-segments, i.e. $f \circ \uparrow seg_B = \uparrow seg_A \circ \exists_f$. In case g preserves implication we have that f is epic iff f preserves the greatest global section.*

Proof. f is epic iff $g \circ f = id_B$ follows from $f = f \circ g \circ f$. If $g \circ f = id_B$ then $f \circ \uparrow seg_B = \uparrow seg_A \circ P(g) = \uparrow seg_A \circ P(g) \circ \exists_g \circ \exists_f \leqslant \uparrow seg_A \circ \exists_f \leqslant f \circ \uparrow seg_B$, i.e. $f \circ \uparrow seg_B = \uparrow seg_A \circ \exists_f$.

Conversely, assume that f preserves $\uparrow$-segments. If $I \in |\underline{\mathbf{E}}|$ and $b \in B$ then as $b \circ g \circ f \leqslant b$ there is $J \in |\underline{\mathbf{E}}|$ and $\exists e \in I$ (e is epic) $\exists a \in A$ such that $e \circ b = a \circ f \leqslant e \circ b \circ g \circ f \leqslant e \circ b$. It follows that $e \circ b = e \circ b \circ g \circ f$ and as e is epic we see that $b = b \circ g \circ f$, i.e. $g \circ f = id_B$.

The last statement follows from Proposition 2.14, 2) and the fact that g preserves the greatest global section as $f \dashv g$. □

Proposition 2.16. *Let (A, B, f, g) be as in Proposition 2.14, then f preserves binary intersection iff*



commutes.

Proof. Apply Lemma 2.1. □

Corollary 2.14. *Let (A, B, f, g) be as in Proposition 2.14. If f is left exact then g preserves implication iff g is monic.*

Proof. If $id_B = g \circ f$ then g must preserve implication as proved by sticking $g \times id_B$ on top of the commutative pentagon in Proposition 2.16.

Conversely, f preserves and g reflects the greatest global section, whence if g preserves implication it follows that g reflects the order relation and therefore g is monic.

□

Let $(H, \wedge, e, \rightarrow)$ be a Heyting algebra in $\underline{\mathbf{E}}$ and let $\alpha_* = ch_H(e) : H \longrightarrow \Omega$, then we readily see that α_* is left exact.

If α^* is a left adjoint of α_* then α^* preserves the greatest global section, i.e. $true \circ \alpha^* = e$, and as $\uparrow \widehat{seg}_H = \rightarrow \circ \alpha_*$ it follows that

$$\forall I \in |\underline{\mathbf{E}}| \quad \forall n \in \Omega \quad \forall x \in H : (n \circ \alpha^* \rightarrow x) \circ \alpha_* = n \Rightarrow x \circ \alpha_*,$$

as this condition is equivalent with $\alpha^* \dashv \alpha_*$, but according to Proposition 2.16 this means that α^* preserves binary intersection.

If H is a complete Heyting algebra then as $\alpha_* = \downarrow seg_H \circ P(e)$ it follows that in this case α^* exists and equals $\exists_e \circ sup_H$.

Let us shortly investigate some of the consequences of the last group of propositions on the morphisms on the internal power objects in $\underline{\mathbf{E}}$.

If $f \in \mathrm{Hom}_{\underline{\mathbf{E}}}(X, Y)$ then by Proposition 2.14, 1) we have that

s) $\exists_f \circ \downarrow seg_{P(Y)} = \downarrow seg_{P(X)} \circ \exists_{\exists_f}$

If f is monic then $\exists_f$ is also monic, and therefore if we evaluate s) on $\ulcorner true_X \urcorner$ we find that

st) $\ulcorner ch_Y(f) \urcorner \circ \downarrow seg_{P(Y)} = \ulcorner ch_{P(Y)}(\exists_f) \urcorner$

which states that $\downarrow seg$ is a "strength" for $\exists$.

In the same way Proposition 2.14, 2) and Proposition 2.14, 3) contain well known information. From Proposition 2.14, 4) we see that "universal quantification along f preserves $false$ iff the image of f is double negation dense in Y".

If $F : P(A) \longrightarrow P(B)$ is sup-preserving, i.e. $\bigcup_A \circ F = \exists_F \circ \bigcup_B$, then as $F = \exists_{\{\,\}_A \circ F} \circ \bigcup_B$, it follows that $F = P(a) \circ \exists_b$ where $\langle a, b \rangle : R \longrightarrow A \times B$ is the relation from A to B which is determined by $\uparrow seg_R = \{\,\}_A \circ F$.

s1) $\ulcorner true_A \urcorner \circ F = \ulcorner true_A \urcorner \circ P(a) \circ \exists_b = \ulcorner true_R \urcorner \circ \exists_b = \ulcorner ch_B(im(b)) \urcorner$.

It follows that $\ulcorner true_A \urcorner \circ F = \ulcorner true_B \urcorner$ iff b is epic.

s2) If b is monic then $\exists_b$ is monic and by Corollary 2.13 $\exists_b$ preserves binary intersection. It follows that F preserves binary intersection.

Conversely, assume that F preserves binary intersection, and let $x, y \in \mathrm{Hom}_{\underline{\mathbf{E}}}(I, R)$ be two elements of R such that $x \circ b = y \circ b$.

Now $x \circ b \Subset x \circ a \circ \{\ \}_A \circ F$ and $y \circ b \Subset y \circ a \circ \{\ \}_A \circ F$, whence

$$\exists J \in \underline{\mathbf{E}} \quad \exists e \in I \ \text{(epi)} \quad \exists z \in R$$

such that $z \Subset e \circ x \circ a \circ \{\ \}_A \circ P(a)$ and $z \Subset e \circ y \circ a \circ \{\ \}_A \circ P(a)$. It follows that $e \circ x \circ a = z \circ a = e \circ y \circ a$. Thus $x \circ a = y \circ a$ as e is epic, and therefore $x = y$ as $\langle a, b\rangle$ is monic. It follows that b is a monomorphism, i.e. F preserves binary intersection iff b is a monomorphism.

s3) In case $F = \exists_g : P(A) \longrightarrow P(B)$ we see that g is a monomorphism iff $\exists_g$ preserves binary intersection.

s4) The morphisms of the form $P(f) : P(A) \longrightarrow P(B)$ are exactly the lrc-morphisms (i.e. left exact and right continuous).

Indeed, if $F = P(a) \circ \exists_b$ is lrc then b is an isomorphism, by s1) and s2), and therefore $F = P(a) \circ P(b_{-1}) = P(b^{-1} \circ a)$.

In particular if F is both sup- and inf-preserving then there exists a uniquely determined $f \in \mathrm{Hom}_{\underline{\mathbf{E}}}(B, A)$ such that $F = P(f)$.

s5) If $F : P(A) \longrightarrow P(B)$ is an order-preserving isomorphism then $F = \exists_g$ where $g : A \longrightarrow B$ is an isomorphism.

The proof of the tripleability theorem which we are now going to establish does not differ essential from that which was discovered independently by R. Paré, [22]. Indeed, the fact that P reflects all coequalizers can be replaced by the fact that P is faithful. Thus if we are primarily interested in the finite colimits we only need to construct $\exists_f$ for f monic, to verify that $\exists_f \circ P(f) = id$ for f monic and to establish the internal Beck condition for pull back diagrams with two opposite faces monic. The existence of finite colimits now follows from the fact that the Eilenberg-Moore category for a monad has the same type of inverse limits as the base category.

The reason that we have not adopted this approach is that we wanted more than the mere existence, namely the elementary description which does not follow directly from the indicated method.

Theorem 2.5. *Let $\underline{\mathbf{E}}$ be an elementary topos then the functor*

1) $$P : \underline{\mathbf{E}}^{op} \longrightarrow \underline{\mathbf{E}}$$

is tripleable.

Proof. The contravariant functor $P : \underline{\mathbf{E}} \longrightarrow \underline{\mathbf{E}}$ is adjoint to itself on the right, i.e. the covariant functor $P : \underline{\mathbf{E}}^{op} \longrightarrow \underline{\mathbf{E}}$ has a left adjoint. $\underline{\mathbf{E}}^{op}$ has all finite colimits as $\underline{\mathbf{E}}$ has all finite limits.

The functor $P : \underline{\mathbf{E}}^{op} \longrightarrow \underline{\mathbf{E}}$ reflects coequalizers. Indeed, let

2) $$A \xrightarrow{\;i\;} B \underset{g}{\overset{f}{\rightrightarrows}} C$$

be a diagram in $\underline{\mathbf{E}}$ and assume that

$$P(C) \underset{P(g)}{\overset{P(f)}{\rightrightarrows}} P(B) \xrightarrow{P(i)} \!\!\!\!\rightarrow P(A)$$

is a coequalizer diagram in $\underline{\mathbf{E}}$.

As P is faithful we see that $i \circ f = i \circ g$, and as $P(i)$ is epic we know that i is monic. Let $h \in \mathrm{Hom}_{\underline{\mathbf{E}}}(X, B)$ and assume that $h \circ f = h \circ g$. Then $P(f) \circ P(h) = P(g) \circ P(h)$ and therefore there exists $q \in \mathrm{Hom}_{\underline{\mathbf{E}}}(P(A), P(X))$ such that $P(i) \circ q = P(h)$. As $P(i)$ is epi (split) it follows that q is an internal functor which is sup- and inf-preserving. It follows from s4) that there exists $m \in \mathrm{Hom}_{\underline{\mathbf{E}}}(X, A)$ such that $q = P(m)$. Again, as P is faithful, we see that $h = m \circ i$ proving that 2) is an equalizer diagram.

The functor $P : \underline{\mathbf{E}}^{op} \longrightarrow \underline{\mathbf{E}}$ preserves coequalizers of P-contractible pairs. Indeed let

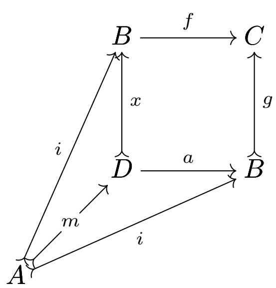
3) $$A \rightarrowtail^{\!i} B \underset{g}{\overset{f}{\rightrightarrows}} C$$

be an equalizer such that $P(f)$ and $P(g)$ have a contractible coequalizer in $\underline{\mathbf{E}}$, i.e.

$$P(C) \underset{s}{\overset{P(f),\,P(g)}{\rightleftarrows}} P(B) \underset{t}{\overset{q}{\rightleftarrows}} Q$$

such that $P(f) \circ q = P(g) \circ q$, $t \circ q = id_Q$, $s \circ P(g) = id_{P(B)}$ and $s \circ P(f) = q \circ t$. We claim that $P(i) = coeq(P(f), P(g))$.

Consider the diagram

4) 

where the inner square is a pull back and m is the proof of $i \circ f = i \circ g$. Notice that g and x are monic as $P(g)$ is epi (split by s).

We claim that $a \circ g = a \circ f$. Indeed $P(a \circ f) = P(f) \circ P(a) = P(f) \circ s \circ P(g) \circ P(a) = P(f) \circ s \circ P(f) \circ P(x) = P(f) \circ q \circ t \circ P(x) = P(g) \circ q \circ t \circ P(x) = P(g) \circ s \circ P(f) \circ P(x) = P(g) \circ s \circ P(g) \circ P(a) = P(g) \circ P(a) = P(a \circ g)$, and so $a \circ f = a \circ g$ as P is faithful.

Let $h \in \mathrm{Hom}_{\underline{\mathbf{E}}}(D, A)$ be the proof of $a \circ f = a \circ g$, i.e. h is uniquely determined by the equation $h \circ i = a$. Notice that $m \circ h \circ i = m \circ a = i$. It follows that $m \circ h = id_A$.

We claim that

$$P(C) \underset{\exists_g}{\overset{P(f),\,P(g)}{\rightleftarrows}} P(B) \underset{P(h) \circ \exists_x}{\overset{P(i)}{\rightleftarrows}} P(A)$$

is a contractible coequalizer diagram in $\underline{\mathbf{E}}$.

i) $P(f) \circ P(i) = P(g) \circ P(i)$ as $i \circ f = i \circ g$.

ii) $P(h) \circ \exists_x \circ P(i) = P(h) \circ \exists_x \circ P(m \circ x) = P(h) \circ \exists_x \circ P(x) \circ P(m) =$

$P(h) \circ P(m) = P(m \circ h) = P(id_A) = id_{P(A)}$ as x is monic and $m \circ h = id_A$.

iii) $\exists_g \circ P(g) = id_{P(B)}$ as g is monic.

iv) $\exists_g \circ P(f) = P(a) \circ \exists_x = P(h \circ i) \circ \exists_x = P(i) \circ P(h) \circ \exists_x$, where the first equality is a consequence of the internal Beck condition applied to the diagram 4).

This concludes the proof of Theorem 2.5. □

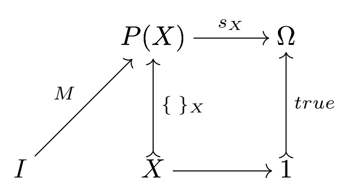
One of the important operations in elementary topoi which is not directly comprised in the calculus of $\exists$, P and $\forall$ is **the unique existentiation**. This concept was introduced in elementary topoi by P. Freyd in **Aspects of Topoi**. It is used whenever for a given $f \in \mathrm{Hom}_{\underline{\mathbf{E}}}(A, B)$ we need to describe the subobject of the domain A of f to which the restriction of f defines an isomorphism onto the corresponding image. We shall use the unique existentiation in its internal form

$$f : A \longrightarrow B \qquad \text{into} \qquad \exists!_f : P(A) \longrightarrow P(B)$$

as well as its internal strength:

$$\boldsymbol{\exists!}_{A,B} : B^A \longrightarrow P(B)^{P(A)}$$

Consider the following pull back diagram:



What does it mean to say that M is a singleton?, i.e. that $M \circ s_X = true_I$.

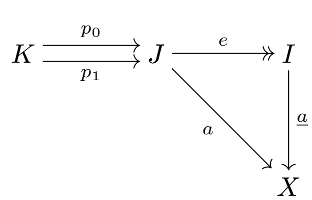
We have the following three equivalent elementary descriptions:

1) $\exists a : I \longrightarrow X$ such that $M = a \circ \{\ \}_X$.

2) $\exists a : I \longrightarrow X$ such that $a \underline{\in} M$ and $\forall J \in |\underline{\mathbf{E}}|\ \forall i \in I\ \forall x \in X : x \underline{\in} i \circ M$ implies $x = i \circ a$.

3) $\exists J \in |\underline{\mathbf{E}}|\ \exists e \in I$ (epi) $\exists a \in X$ such that $a \underline{\in} e \circ M$ and $\forall K \in |\underline{\mathbf{E}}|\ \forall i \in I\ \forall x, y \in X$: $x \underline{\in} i \circ M$ and $y \underline{\in} i \circ M$ implies $x = y$.

Clearly 1. ⇔ 2. ⇒ 3. If 3. is valid we may prove 2. as follows:



Let p_0, p_1 be the kernel-pair of e.Let $i = p_0 \circ e$ $(= p_1 \circ e)$, and as $p_0 \circ a \underline{\in} i \circ M$ and $p_1 \circ a \underline{\in} i \circ M$ we have that $p_0 \circ a = p_1 \circ a$. As $e = coeq(p_0, p_1)$ there exists $\underline{a} \in \mathrm{Hom}_{\underline{\mathbf{E}}}(I, X)$ uniquely determined by $e \circ \underline{a} = a$.

This proves the validity of 2.

Let $T_X : P(X) \longrightarrow PP(X)$ be the exponential adjoint of $\wedge_{P(X)} \circ s_X$. If $f \in \mathrm{Hom}_{\underline{\mathbf{E}}}(X, Y)$ we define $\exists!_f : P(X) \longrightarrow P(Y)$ by the equation

$$\exists!_f = P(X) \xrightarrow{T_X} PP(X) \xrightarrow{PP(f)} PP(Y) \xrightarrow{P(\{\ \}_Y)} P(Y)$$

$\exists!_f$ is called the internal unique existentiation along f, and it has the following elementary description:

$$\begin{array}{ccc} P(X) & \xrightarrow{\exists!_f} & P(Y) \\ {\scriptstyle M}\uparrow & & \\ I & \xrightarrow[y]{} & Y \end{array}$$

$\forall I \in |\underline{\mathbf{E}}| \quad \forall M \in P(X) \quad \forall y \in Y : y \underline{\in} M \circ \exists!_f$ iff $y \circ \{\ \}_Y \circ P(f) \underline{\in} M \circ T_X$ iff there exists exactly one $x \in \mathrm{Hom}_{\underline{\mathbf{E}}}(I, X)$ such that $x \circ f = y$ and $x \underline{\in} M$.

The strength of the unique existentiation is given by its exponential adjoint

$$\widehat{\exists!_{X,Y}} = \{\ \}_{Y^X} \times id_{P(X)} \circ \tilde{p}_{Y^X,X} \circ \exists!_{ev_{X,Y}} : Y^X \times P(X) \longrightarrow P(Y)$$

and has the following elementary description:

$$\begin{array}{ccc} Y^X \times P(X) & \xrightarrow{\widehat{\exists!_{X,Y}}} & P(Y) \\ \uparrow{\scriptstyle \langle g, A\rangle} & & \\ I & \xrightarrow[y]{} & Y \end{array}$$

$\forall I \in |\underline{\mathbf{E}}| \quad \forall g \in Y^X \quad \forall A \in P(X) \quad \forall y \in Y : y \underline{\in} \langle g, A\rangle \circ \widehat{\exists!_{X,Y}}$ iff there exists exactly one $x \in \mathrm{Hom}_{\underline{\mathbf{E}}}(I, X)$ such that $\langle g, x\rangle \circ ev_{X,Y} = y$.

An explanation of the tripleability of $P : \underline{\mathbf{E}}^{op} \longrightarrow \underline{\mathbf{E}}$ can be found in the work of M. Stone, [24], on the characterization of the lattice theoretic structure of power sets. Stone explains it (in the category of sets) to be that of a complete atomic Boolean algebra. This theorem has a topos theoretic version which has its own intrinsic beauty, but which furthermore has interesting applications.

Let $S(\underline{\mathbf{E}})$ be the Stone category of the elementary topos $\underline{\mathbf{E}}$. The objects of $S(\underline{\mathbf{E}})$ are the complete Heyting algebras in $\underline{\mathbf{E}}$, and $\mathrm{Hom}_{S(\underline{\mathbf{E}})}(H, K)$ is the set of Stone morphisms from H to K. Recall that $f \in \mathrm{Hom}_{S(\underline{\mathbf{E}})}(H, K)$ iff $f = f_*$ in the system

$$H \underset{f_*}{\overset{f!}{\underset{\xleftarrow{\ f^*\ }}{\rightrightarrows}}} K$$

where $f! \dashv f^* \dashv f_*$ and f^* preserves implication.

We have seen that the assignment X into $P(X)$ and $f \in \mathrm{Hom}_E(X,Y)$ into $S(f) = (\exists_f, P(f), \forall_f) \in \mathrm{Hom}_{S(\underline{\mathbf{E}})}(P(X), P(Y))$ defines a functor

$$S : \underline{\mathbf{E}} \longrightarrow S(\underline{\mathbf{E}}).$$

We claim that this functor has a right adjoint

$$T : S(\underline{\mathbf{E}}) \longrightarrow \underline{\mathbf{E}},$$

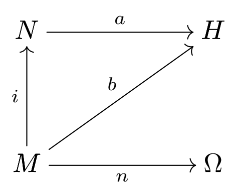
the existence and nature we shall now proceed to describe.

In the category of sets $T(H)$ can be described as the set of atoms in H, where an atom in H is an element $a \in H$ such that $0 \neq a$ and for all $b \in H$ we have that $b < a$ implies $b = 0$. Alternatively, this property may be described by stating that the map

$$q : \{b \in H | b \leqslant a\} \longrightarrow 2$$

which is defined by $(\{1\})q^{-1} = \{a\}$ is an order preserving bijection.

Guided by this observation we introduce the concept of an atom $a : N \longrightarrow H$ in a complete Heyting algebra H in $\underline{\mathbf{E}}$ as follows:



a is an atom in H iff

$\forall M \in |\underline{\mathbf{E}}| \quad \forall i \in N \quad \forall n \in \Omega \quad \exists!\ b \in H$ such that

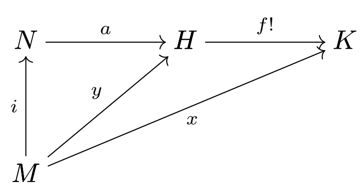
$b \leqslant i \circ a$ and $\langle i \circ a, b\rangle \circ \delta_H = n$ iff

$a \circ at_H = true_N$ where

$$at_H = H \xrightarrow{\langle \{\ \}_H, \downarrow seg_H \rangle} P(H) \times P(H) \xrightarrow{\widehat{\exists !}_{H,\Omega}} P(\Omega) \xrightarrow{\forall_\Omega} \Omega$$

Lemma 2.2. *Stone morphisms preserve atoms.*

Proof. Let $f \in \mathrm{Hom}_{S(\underline{E})}(H, K)$ and consider the following diagram



We shall verify that $\forall M \in |\underline{\mathbf{E}}| \quad \forall i \in N \quad \forall x \in K$ such that $x \leqslant i \circ a \circ f! \quad \exists!\ y \in H$ such that $y \circ f! = x$ and furthermore $\langle i \circ a, y\rangle \circ \delta_H = \langle i \circ a \circ f!, x\rangle \circ \delta_K$.

If $x \leqslant i \circ a \circ f!$ then $x = i \circ a \circ f! \wedge x \underset{(2)}{=} (i \circ a \wedge x \circ f^*) \circ f!$

and

$$\langle i\circ a\circ f!, x\rangle\circ\delta_K \underset{(1)}{=} (i\circ a\circ f! \to x)\circ\alpha_* \underset{(4)}{=} (i\circ a \to x\circ f^*)\circ f_*\circ\alpha_* = (i\circ a \to x\circ f^*)\circ\beta_* = (i\circ a \to (i\circ a\wedge x\circ f^*))\circ\beta_* \underset{(1)}{=} \langle i\circ a, i\circ a\wedge x\circ f^*\rangle\circ\delta_H$$

(1) as the order $\uparrow \widehat{seg}_H = \!\!\to \circ\alpha_*$ on a Heyting algebra (page 36) is antisymmetric, (2) by Proposition 2.14 2), and (4) and by Proposition 2.14 4) and so if we let $y = i\circ a\wedge x\circ f^*$ we have $y \leqslant i\circ a$ and $y\circ f! = x$.

To prove the uniqueness of y consider the order preserving functions

$$\mathrm{Hom}_{\underline{\mathbf{E}}}(\mathrm{M},\mathrm{H}) \underset{L}{\overset{R}{\rightleftarrows}} \mathrm{Hom}_{\underline{\mathbf{E}}}(\mathrm{M},\mathrm{K})$$

given by $(y)R = (i\circ a\to y)\circ f_*$ and $(x)L = i\circ a\wedge x\circ f^*$, then $L\dashv R$ and so $R\circ L\circ R = R$. Thus, if $y\in \mathrm{Hom}_{\underline{\mathbf{E}}}(\mathrm{M},\mathrm{H})$ then $(i\circ a\to ((i\circ a\to y)\circ f_*\circ f^*\wedge i\circ a))\circ f_* = (i\circ a\to y)\circ f_*$ and so if $y\leqslant i\circ a$ we know that $\langle i\circ a, (i\circ a\to y)\circ f_*\circ f^*\wedge i\circ a\rangle\circ\delta_H = \langle i\circ a, y\rangle\circ\delta_H$. It follows that $(i\circ a\to y)\circ f_*\circ f^*\wedge i\circ a = y$ as $i\circ a$ is an atom.

Let $y,z\in H$ such that $y,z\leqslant i\circ a$ then $i\circ a\wedge z\circ f!\circ f^*\leqslant y \Leftrightarrow z\circ f!\circ f^*\leqslant i\circ a\to y \Leftrightarrow z\circ f!\leqslant (i\circ a\to y)\circ f_* \Leftrightarrow z\leqslant (i\circ a\to y)\circ f_*\circ f^* \Leftrightarrow z\leqslant (i\circ a\to y)\circ f_*\circ f^*\wedge i\circ a \Leftrightarrow z\leqslant y$. It follows that $i\circ a\wedge z\circ f!\circ f^* = z$.

Let $y,z\in H$ such that $y,z\leqslant i\circ a$ and assume that $y\circ f! = z\circ f!$ then $z = y$.
Indeed $z = i\circ a\wedge z\circ f!\circ f^* = i\circ a\wedge y\circ f!\circ f^* = y$.

Finally, let $M\in|\underline{\mathbf{E}}|$, $i\in N$ and $n\in\Omega$ and let $y_0\in H$ be the unique solution to $y_o\leqslant i\circ a$ and $\langle i\circ a, y_0\rangle\circ\delta_H = n$.

Let $x_0 = y_0\circ f!$. As $x_0 = y_0\circ f!\leqslant i\circ a\circ f!$ we know $\exists! y\in H$ such that $y\leqslant i\circ a$ and $y\circ f! = x_0$ and for this y we know that $\langle i\circ a, y\rangle\circ\delta_H = \langle i\circ a\circ f!, x_0\rangle\circ\delta_K$.

As $y_0, y\leqslant i\circ a$ and $y\circ f! = x_0 = y_0\circ f!$ we have that $y_0 = y$ proving that $\langle i\circ a\circ f!, x_0\rangle\circ\delta_K = \langle i\circ a, y\rangle\circ\delta_H = \langle i\circ a, y_0\rangle\circ\delta_H = n$.

As for the uniqueness, assume that $x_1\leqslant i\circ a\circ f!$ and $\langle i\circ a\circ f!, x_1\rangle\circ\delta_K = n$. Let $y_1\leqslant i\circ a$ be the unique solution to $y_1\circ f! = x_1$. As y_1 satisfies $\langle i\circ a, y_1\rangle\circ\delta_H = \langle i\circ a\circ f!, x_1\rangle\circ\delta_K = n$ we see that $y_1 = y_0$ as a is an atom, whence $x_1 = y_1\circ f! = y_0\circ f! = x_0$.

This proves that $a\circ f!$ is an atom. □

Let $H\in|S(\underline{\mathbf{E}})|$ and let $i_H : T(H)\longrightarrow H$ be the extension of the atoms in H, i.e. the subobject of H defined by the following pull back diagram:

$$\begin{array}{ccc} H & \xrightarrow{\ at_H\ } & \Omega \\ \uparrow{\scriptstyle i_H} & & \uparrow{\scriptstyle true} \\ T(H) & \longrightarrow & 1 \end{array}$$

By Lemma 2.2 we see that there exists for any $f\in \mathrm{Hom}_{S(\underline{\mathbf{E}})}(H,K)$ a morphism $T(f) : T(H)\longrightarrow T(K)$ uniquely determined by the condition

$$i_H\circ f! = T(f)\circ i_K$$

Consequently, the assignment

$$H \text{ into } T(H) \text{ and } f \in \mathrm{Hom}_{S(\underline{\mathbf{E}})}(H,K) \text{ into } T(f) \in \mathrm{Hom}_{\underline{\mathbf{E}}}(T(H),T(K))$$

defines a functor

$$T : S(\underline{\mathbf{E}}) \longrightarrow \underline{\mathbf{E}}.$$

Let $X \in |\underline{\mathbf{E}}|$ and $H \in |S(\underline{\mathbf{E}})|$ and

$$P(X) \underset{g^*}{\overset{g!}{\rightleftarrows}} H \qquad g! \dashv g^*$$

a pair of internal adjoint functors. By Theorem 2.1 we know that if $h = \{\ \}_X \circ g!$ then $g! = \exists_h \circ sup_H$ and $g^* = \downarrow seg_H \circ P(h)$.

Observe that as g^* preserves binary intersection g^* preserves implication iff

$$\forall I \in |\underline{\mathbf{E}}| \quad \forall a,b \in H \quad \forall x \in X : x \circ \{\ \}_X \wedge a \circ g^* \leqslant b \circ g^* \text{ implies } x \circ h \wedge a \leqslant b.$$

As g^* is an internal functor on complete lattices then g^* has a right adjoint iff g^* is sup-preserving iff $\forall I \in |\underline{\mathbf{E}}| \quad \forall A \in P(H) \quad \forall x \in X : x \circ h \leqslant A \circ sup_H$ implies $\exists J \in |\underline{\mathbf{E}}| \quad \exists\, e \in I$ (epi) $\exists\, a \in H$ such that $a \underline{\in} e \circ A$ and $e \circ x \circ h \leqslant a$.

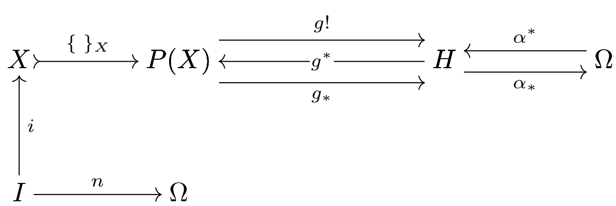
Lemma 2.3. *The internal functor $g^* = \downarrow seg_H \circ P(h)$ from H to $P(X)$ preserves implication and has a right adjoint iff the morphism $h \in \mathrm{Hom}_{\underline{\mathbf{E}}}(X,H)$ is an atom in H.*

Proof. Assume that $g = (g!, g^*, g_*) \in \mathrm{Hom}_{S(\underline{\mathbf{E}})}(P(X), H)$ and let $N \in |\underline{\mathbf{E}}|$, $n \in X$, $a,b \in H$ such that $a \leqslant n \circ h$, $b \leqslant n \circ h$ and $\langle n \circ h, a\rangle \circ \delta_H = \langle n \circ h, b\rangle \circ \delta_H$.

If $M \in |\underline{\mathbf{E}}|$, $i \in N$ and $m \in X$ and $m \underline{\in} i \circ (n \circ \{\ \}_X \wedge a \circ g^*)$, i.e. if $m = i \circ n$ and $m \circ h \leqslant i \circ a$ then $i \circ n \circ h = i \circ a$, then $i \circ n \circ h = i \circ b$ and so $m \underline{\in} i \circ b \circ g^*$. As g^* preserves implication we have that $a = n \circ h \wedge a \leqslant b$. Dually $b \leqslant a$, i.e. $a = b$.

This shows that h satisfies the uniqueness property of atoms.

Next, consider the following diagram



where $I \in |\underline{\mathbf{E}}|$, $i \in X$ and $n \in \Omega$.

As $i \circ h \wedge n \circ \alpha^* \leqslant i \circ h$ and as $\langle i \circ h, i \circ h \wedge n \circ \alpha^*\rangle \circ \delta_H = (i \circ h \to (i \circ h \wedge n \circ \alpha^*)) \circ \alpha_* = (i \circ h \to n \circ \alpha^*) \circ \alpha_* = (i \circ \{\ \}_X \circ g! \to n \circ \alpha^*) \circ \alpha_* = (i \circ \{\ \}_X \Rightarrow_{P(X)} n \circ \alpha^* \circ g^*) \circ g_* \circ \alpha_* = (i \circ \{\ \}_X \Rightarrow_{P(X)} n \circ \Delta_X) \circ \forall_X = i \circ \{\ \}_X \circ \exists_X \Rightarrow_{P(X)} n = true_I \Rightarrow_{P(X)} n = n$ by repeated applications of Proposition 2.14, we see that h is an atom in H.

Conversely, assume that $h \in \mathrm{Hom}_{\underline{\mathbf{E}}}(X,H)$ is an atom in H. We claim that g^* preserves implication.

Let $I \in |\underline{\mathbf{E}}|$, $a, b \in H$ and $x \in X$ such that $x \circ \{\ \}_X \wedge a \circ g^* \leqslant b \circ g^*$. As $x \circ h \wedge a \wedge b \leqslant x \circ h \wedge a \leqslant x \circ h$ we have that $\langle x \circ h, x \circ h \wedge a \rangle \circ \delta_H \geqslant \langle x \circ h, x \circ h \wedge a \wedge b \rangle \circ \delta_H$. Let $J \in |\underline{\mathbf{E}}|$ and $i \in I$ such that $i \circ x \circ h = i \circ x \circ h \wedge i \circ a$, then $i \circ x \Subset i \circ (x \circ \{\ \}_X \wedge a \circ g^*)$ and so $i \circ x \Subset i \circ b \circ g^*$, i.e. $i \circ x \circ h \leqslant i \circ b$. It follows that $i \circ x \circ h = i \circ x \circ h \wedge i \circ a \wedge i \circ b$ and so $\langle x \circ h, x \circ h \wedge a \rangle \circ \delta_H = \langle x \circ h, x \circ h \wedge a \wedge b \rangle \circ \delta_H$. As $a \circ h$ is an atom in H we conclude that $x \circ h \wedge a = x \circ h \wedge a \wedge b$ and therefore $x \circ h \wedge a \leqslant b$.

This proves that g^* preserves implication.

Finally, to see that g^* has a right adjoint let $I \in |\underline{\mathbf{E}}|$, $A \in P(H)$ and $x \in X$ such that $x \circ h \leqslant A \circ sup_H$.

Let $n \in \mathrm{Hom}_{\underline{\mathbf{E}}}(I, \Omega)$ be the following character:

$$n = \langle x \circ h \circ \{\ \}_H, A \rangle \circ \tilde{p}_{H,H} \circ \exists_{(\to \circ \alpha_*)} \circ sup_\Omega.$$

We claim that $n = true_I$.

As $x \circ h$ is an atom there exists $z \in \mathrm{Hom}_{\underline{\mathbf{E}}}(I, H)$ such that $z \leqslant x \circ h$ and $(x \circ h \to z) \circ \alpha_* = \langle x \circ h, z \rangle \circ \delta_H = n$. It follows that $\langle x \circ h \circ \{\ \}_H, A \rangle \circ \tilde{p}_{H,H} \leqslant (x \circ h \to z) \circ \alpha_* \circ {\downarrow} seg_\Omega \circ P(\to \circ \alpha_*)$.

If $J \in |\underline{\mathbf{E}}|$, $i \in I$ and $a \in H$ such that $a \Subset i \circ A$ then $(i \circ x \circ h \to (i \circ x \circ h \wedge a)) \circ \alpha_* = (i \circ x \circ h \to a) \circ \alpha_* \leqslant (i \circ x \circ h \to i \circ z) \circ \alpha_*$, and so $i \circ x \circ h \wedge a \leqslant i \circ z$ as the map $(i \circ x \circ h \to (\ \ )) \circ \alpha_*$ is injective on the $\downarrow$-segments of $i \circ x \circ h$ in $\mathrm{Hom}_{\underline{\mathbf{E}}}(J, H)$ and as this map preserves binary intersection. It follows that $\langle x \circ h \circ \{\ \}_H, A \rangle \circ \tilde{p}_{H,H} \leqslant z \circ {\downarrow} seg_H \circ P(\wedge)$ and therefore $z \geqslant \langle A, x \circ h \circ \{\ \}_H \rangle \circ \tilde{p}_{H,H} \circ \exists_\wedge \circ sup_H = \langle A, x \circ h \rangle \circ (sup_H \times id_H) \circ \wedge = x \circ h$, i.e $z = x \circ h$ or equivalently $n = true_I$.

Recalling the construction of n we see that $\exists N \in |\underline{\mathbf{E}}|$, $\exists e \in I$ (epi), $\exists a \in H$ such that $(e \circ x \circ h \to a) \circ \alpha_* = true_N$ i.e. $e \circ x \circ h \leqslant a$.

This concludes the proof of Lemma 2.3. □

Lemma 2.4. *Compatible atoms are equal.*

Proof. Let $H \in |S(\underline{\mathbf{E}})|$ and $a, b \in \mathrm{Hom}_{\underline{\mathbf{E}}}(N, H)$ be two atoms in H such that $a \leqslant b$.

$\forall I \in |\underline{\mathbf{E}}| \quad \forall y \in H \quad \exists!\ x \in H$ such that $\langle i \circ b, y \rangle \circ \delta_H = \langle i \circ a, x \rangle \circ \delta_H$ and $x \leqslant i \circ a$.

Now $\langle i \circ a, x \rangle \circ \delta_H = (i \circ a \to x) \circ \alpha_* = (i \circ b \to ((i \circ a \to x) \wedge i \circ b)) \circ \alpha_* = \langle i \circ b, (i \circ a \to x) \wedge i \circ b \rangle \circ \delta_H$, whence $\langle i \circ b, y \rangle \circ \delta_H = \langle i \circ b, (i \circ a \to x) \wedge i \circ b \rangle \circ \delta_H$.

Taking $I = N$, $i = id_N$ and $y = a$ we get that $\langle b, a \rangle \circ \delta_H = \langle b, (a \to x) \wedge b \rangle \circ \delta_H$ and as $a \leqslant b$ and $(a \to x) \wedge b \leqslant b$, we conclude that $a = (a \to x) \wedge b$ as b is an atom. But $a \leqslant a \to x$ iff $a \leqslant x$ iff $a \to x = e_N$, whence $a = e_N \wedge b = b$.

This concludes the proof of Lemma 2.4. □

Corollary 2.15. *If* $H \in |S(\underline{\mathbf{E}})|$ *then* $i_H \circ {\downarrow} seg_H \circ P(i_H) = \{\ \}_{T(X)}$.

Proposition 2.17. $\underline{\mathbf{E}}$ *is a coreflective subcategory of the Stone category* $S(\underline{\mathbf{E}})$.

Proof. Let $H \in |S(\underline{\mathbf{E}})|$. By Lemma 2.3 we have a Stone morphism $f = (f!, f^*, f_*) : PT(H) \longrightarrow H$ determined by $\{\ \}_{T(H)} \circ f! = i_H$.

By Lemma 2.4 we have that $\{\ \}_{T(H)} \circ f! \circ f^* = \{\ \}_{T(H)} \circ \exists_{i_H} \circ sup_H \circ {\downarrow} seg_H \circ P(i_H) = i_H \circ {\downarrow} seg_H \circ P(i_H) = \{\ \}_{T(H)}$. It follows that $f! \circ f^* = id_{PT(H)}$ as f^* is sup-preserving.

If $g \in \mathrm{Hom}_{S(\underline{\mathbf{E}})}(P(X), H)$ then there exists, by Lemma 2.3, a unique $\underline{h} \in \mathrm{Hom}_{\underline{\mathbf{E}}}(X, T(H))$ such that $\underline{h} \circ i_H = \{\ \}_X \circ g!$. It follows that $S(\underline{h}) \circ f = g$.

This proves that $S \dashv T$. □

From the characterization of the internal substitution functors it follows that the front adjunction for $S \dashv T$ is a natural isomorphism, i.e. the atoms of an internal power object are the singletons. This can also be seen by applying Lemma 2.3, Lemma 2.4 and Proposition 1.3.

We shall say that an object $H \in |S(\underline{\mathbf{E}})|$ is **atomic** iff the sup of the atoms in H is the greatest global section e in H.

It is now a formal consequence of Proposition 2.17 and Proposition 2.15 that the objects in $S(\underline{\mathbf{E}})$ for which the end adjunction $f : PT(H) \longrightarrow H$ is an isomorphism are exactly those for which $true_{T(H)} \circ f! = e$. This means that we have

Theorem 2.6 (Stone). *The objects of the form $P(X)$ are the complete atomic Heyting algebras in $\underline{\mathbf{E}}$.*

Chapter 3

Functors on Elementary Topoi

In this chapter we shall study some of the intrinsic properties of the functors used in comparing elementary topoi. The type of the results and the method of proof requires a systematic notation which we shall presently develop.

Let

$$F : \underline{\mathbf{E}} \longrightarrow \underline{\mathbf{E}}_0$$

be a functor on elementary topoi and assume that F preserves all binary cartesian products and the terminal object. The functor F comes equipped with

1) a natural isomorphism

$$\tilde{F} = \{\tilde{F}_{A,B} : F(A) \times F(B) \longrightarrow F(A \times B)\}_{(A,B)\in|\underline{\mathbf{E}}|\times|\underline{\mathbf{E}}|}$$

2) a natural transformation

$$\hat{F} = \{\hat{F}_{A,B} : F(B^A) \longrightarrow F(B)^{F(A)}\}_{(A,B)\in|\underline{\mathbf{E}}|\times|\underline{\mathbf{E}}|}$$

3) a canonical isomorphism

$$F^0 : 1_0 \longrightarrow F(1),$$

making the system $(F, \tilde{F}, \hat{F}, F^0)$ into a cartesian closed functor. Finally we shall be using the character

4) $d = ch_{F(\Omega)}(F(true)) : F(\Omega) \longrightarrow \Omega_0$.

Definition 3.1. *A Functor $F : \underline{\mathbf{E}} \longrightarrow \underline{\mathbf{E}}_0$ of elementary topoi is said to be* <u>*logical*</u> *iff*

1) F is left exact.

2) F preserves exponentiation (i.e. $\hat{F}$ is a natural isomorphism).

3) F preserves the subobject classifier (i.e. d is an isomorphism).

Logical functors are very important. Indeed, as they preserve all the axioms characterizing an elementary topos, it follows that all constructions in elementary topoi based on these axioms will be preserved by logical functors. In particular, logical functors are right exact.

Let

$$\underline{\mathbf{E}} \underset{R}{\overset{L}{\rightleftarrows}} \underline{\mathbf{E}}_0\,, \qquad L \dashv R$$

be a pair of adjoint functors on elementary topoi.We shall deal with this adjunction in terms of the two natural transformations:

1. the front adjunction

   $t = \{t_A : A \longrightarrow RL(A)\}_{A\in|\underline{\mathbf{E}}|}$

2. the end adjunction

   $v = \{v_X : LR(X) \longrightarrow X\}_{X\in|\underline{\mathbf{E}}_0|}$

The functor R is left exact and therefore it has the structure $(R, \tilde{R}, \hat{R}, R^0)$ described above. The character $ch_{R(\Omega_0)}(R(true_0))$ will be denoted s. Without further assumptions on L we only have the character $ch_{L(\Omega)}(L(true))$ which will be denoted d. Observe that L preserves binary cartesian products iff

$$\{R(X^{L(A)}) \xrightarrow{\hat{R}_{L(A),X}} R(X)^{RL(A)} \xrightarrow{R(X)^{t_A}} R(X)^A\}_{(A,X)\in|\underline{\mathbf{E}}|\times|\underline{\mathbf{E}}_0|}$$

is a natural isomorphism.

Definition 3.2. *A geometric functor from $\underline{\mathbf{E}}_0$ to $\underline{\mathbf{E}}$ is a pair of adjoint functors*

$$\underline{\mathbf{E}} \underset{R}{\overset{L}{\rightleftarrows}} \underline{\mathbf{E}}_0\,, \qquad L \dashv R$$

on elementary topoi, where the left adjoint is left exact.

In accordance with the terminology from sheaf theory:

the left adjoint L is called the inverse image functor and

the right adjoint R is called the direct image functor.

Notice that the direction of a geometric functor is that of the right adjoint R.

Definition 3.3. *An essential functor from $\underline{\mathbf{E}}_0$ to $\underline{\mathbf{E}}$ is a geometric functor from $\underline{\mathbf{E}}_0$ to $\underline{\mathbf{E}}$ where the inverse image functor has a left adjoint*

$$\underline{\mathbf{E}} \;\overset{T}{\longleftarrow}\;\overset{L}{\longrightarrow}\;\underset{R}{\longleftarrow}\; \underline{\mathbf{E}}_0 \qquad T \dashv L \dashv R$$

We shall deal with the adjunction $T \dashv L$ in terms of the two natural transformations:

1. the front adjunction

   $\eta = \{\eta_X : X \longrightarrow LT(X)\}_{X\in|\underline{\mathbf{E}}_0|}$

2. the end adjunction

   $\epsilon = \{\epsilon_A : TL(A) \longrightarrow A\}_{A \in |\underline{\mathbf{E}}|}$

   In the case of essential functors we shall use the natural transformation

3. $\theta = \{\theta_{X,A} : A^{T(X)} \longrightarrow R(L(A)^X)\}_{(A,X) \in |\underline{\mathbf{E}}| \times |\underline{\mathbf{E}}_0|}$

θ is the conjugate of the natural transformation $\hat{L}$ and θ and $\hat{L}$ are defined in terms of each other by the following diagrams:

$$\begin{array}{ccc} L(A^{T(X)}) & \xrightarrow{L(\theta_{X,A})} & LR(L(A)^X) \\ \Big\downarrow{\scriptstyle \hat{L}_{T(X),A}} & & \Big\downarrow{\scriptstyle v_{L(A)^X}} \\ L(A)^{LT(X)} & \xrightarrow{L(A)^{\eta_X}} & L(A)^X \end{array}$$

and

$$\begin{array}{ccc} B^A & \xrightarrow{t_{B^A}} & RL(B^A) \\ \Big\downarrow{\scriptstyle B^{\epsilon_A}} & & \Big\downarrow{\scriptstyle R(\hat{L}_{A,B})} \\ B^{TL(A)} & \xrightarrow{\theta_{L(A),B}} & R(L(B)^{L(A)}) \end{array}$$

Notice that θ is pointwise monic (iso) iff $\hat{L}$ is pointwise monic (iso).

Definition 3.4. *A local homeomorphism from $\underline{\mathbf{E}}_0$ to $\underline{\mathbf{E}}$ is an essential functor from $\underline{\mathbf{E}}_0$ to $\underline{\mathbf{E}}$ where the inverse image functor is logical.*

$$\underline{\mathbf{E}} \;\underset{\underset{R}{\longleftarrow}}{\overset{\overset{T}{\longleftarrow}}{\xrightarrow{\;L\;}}}\; \underline{\mathbf{E}}_0 \qquad T \dashv L \dashv R$$

and L is logical.

Let

$$\underline{\mathbf{E}} \;\underset{R}{\overset{L}{\rightleftarrows}}\; \underline{\mathbf{E}}_0\ , \qquad L \dashv R$$

be a pair of adjoint functors on elementary topoi such that the left adjoint L preserves binary cartesian products and the terminal object. Under these assumptions we know that the morphism

$$\hat{d} = \Omega \xrightarrow{t_\Omega} RL(\Omega) \xrightarrow{R(d)} R(\Omega_0)$$

is a left exact morphism of bounded lattices in $\underline{\mathbf{E}}$.

Indeed, R preserves all such structures as R is left exact, and as the lower semilattice structure of the subobject classifier is equationally equivalent with that of a commutative monoid whose multiplication is idempotent, it follows that L must preserve not only this structure, but also the subobject of $\Omega \times \Omega$ defining the associated order relation. It follows that t_Ω is left exact as t is a natural transformation. Finally d is left exact as $L(true)$ is the greatest global section in the induced ordering on $L(\Omega)$ and $d = ch_{L(\Omega)}(L(true))$.

Our first theorem in this chapter is due to W. Mitchell [19]. It is included here, not because the above observations yield the refinement that d and $\hat{d}$ are always left exact, but because we want to stress the essential feature (namely $\hat{d} \dashv s$) of this theorem. Indeed, this adjunction is important for any closer analysis of geometric functors on elementary topoi.

Theorem 3.1. *Let*

$$\underline{\mathbf{E}} \underset{R}{\overset{L}{\rightleftarrows}} \underline{\mathbf{E}}_0 \,, \qquad L \dashv R$$

be a pair of adjoint functors on elementary topoi and assume that the left adjoint L preserves binary cartesian products and the terminal object. Then the following statements are equivalent:

1) L is left exact.

2) L preserves the pull back diagram defining s.

3) $\hat{d} \dashv s$.

Proof.
1) $\Rightarrow$ 2) This is obviously the case.
2) $\Rightarrow$ 3) We know that both $\hat{d}$ and s are internal functors, and as $true \circ \hat{d} \circ s = true$ we have that $id_\Omega \leqslant \hat{d} \circ s$. By 2) we know that $L(s) \circ d = ch_{LR(\Omega_0)}(LR(true_0))$, and therefore we see that $L(s) \circ d \leqslant v_{\Omega_0}$ by the naturality of v. It follows that

$$s \circ \hat{d} = t_{R(\Omega_0)} \circ R(L(s) \circ d) \leqslant t_{R(\Omega_0)} \circ R(v_{\Omega_0}) = id_{R(\Omega_0)}.$$

This proves that $\hat{d} \dashv s$.

3) $\Rightarrow$ 1) Let $i : A \longrightarrow B$ be a monomorphism in $\underline{\mathbf{E}}$ such that $L(i)$ is monic. As t is natural we have that

$$ch_B(i) \leqslant t_B \circ R(ch_{L(B)}(L(i))) \circ s,$$

and as $\hat{d} \dashv s$ we deduce that

$$t_B \circ R(L(ch_B(i)) \circ d) = ch_B(i) \circ \hat{d} \leqslant t_B \circ R(ch_{L(B)}(L(i))),$$

i.e.

$$L(ch_B(i)) \circ d \leqslant ch_{L(B)}(L(i)).$$

As L is a functor we have that

$$ch_{L(B)}(L(i)) \leqslant L(ch_B(i)) \circ d$$

i.e.

$$ch_{L(B)}(L(i)) = L(ch_B(i)) \circ d.$$

In particular if $i = \triangle_A : A \longrightarrow A \times A$ we get that

$$\delta_{L(A)} = \tilde{L}_{A,A} \circ L(\delta_A) \circ d.$$

Recall that a morphism $i : A \longrightarrow B$ is monic iff $i \times i \circ \delta_B = \delta_A$. It follows, therefore, from the below calculation that L preserves monomorphisms. Indeed, if $i : A \longrightarrow B$ is a monomorphism in $\underline{\mathbf{E}}$, then

$$L(i) \times L(i) \circ \delta_{L(B)} = L(i) \times L(i) \circ \tilde{L}_{B,B} \circ L(\delta_B) \circ d =$$

$$\tilde{L}_{A,A} \circ L(i \times i) \circ L(\delta_B) \circ d = \tilde{L}_{A,A} \circ L(i \times i \circ \delta_B) \circ d =$$

$$\tilde{L}_{A,A} \circ L(\delta_A) \circ d = \delta_{L(A)}.$$

Finally, if $f, g \in \mathrm{Hom}_{\underline{\mathbf{E}}}(B, C)$ then

$$ch_{L(B)}(eq(L(f), L(g))) = \langle L(f), L(g) \rangle \circ \delta_{L(C)} =$$

$$\langle L(f), L(g) \rangle \circ \tilde{L}_{C,C} \circ L(\delta_C) \circ d = L(\langle f, g \rangle) \circ L(\delta_C) \circ d =$$

$$L(\langle f, g \rangle \circ \delta_C) \circ d = L(ch_B(eq(f, g))) \circ d.$$

This proves that L preserves equalizers and concludes the proof of Theorem 3.1. □

Corollary 3.1. *Let*

$$\underline{\mathbf{E}} \underset{R}{\overset{L}{\rightleftarrows}} \underline{\mathbf{E}}_0 , \qquad L \dashv R$$

be a geometric functor from $\underline{\mathbf{E}}_0$ to $\underline{\mathbf{E}}$, then the following statements are equivalent:

1) s is epic.

2) $\hat{d}$ is monic.

3) t_Ω is monic.

4) t is a pointwise monic natural transformation.

5) L is a faithful functor.

6) $j = \hat{d} \circ s = id_\Omega$.

Proof. Clearly 5) $\iff$ 4) $\Rightarrow$ 3), and as $\hat{d} \dashv s$ we have that
1) $\iff$ 2) $\iff$ 6) $\Rightarrow$ 3).

3) $\Rightarrow$ 6): Consider the diagram

$$\begin{array}{ccccccc}
\Omega & \xrightarrow{t_\Omega} & RL(\Omega) & \xrightarrow{R(d)} & R(\Omega_0) & \xrightarrow{s} & \Omega \\
\uparrow{\scriptstyle true} \quad * & & \uparrow{\scriptstyle RL(true)} \quad ** & & \uparrow{\scriptstyle R(true_0)} \quad *** & & \uparrow{\scriptstyle true} \\
1 & \underset{\sim}{\xrightarrow{t_1}} & RL(1) & \underset{\sim}{\longrightarrow} & R(1_0) & \underset{\sim}{\longrightarrow} & 1
\end{array}$$

$**$ and $***$ are pull back diagrams and $*$ is a commutative square by the naturality of t. If t_Ω is monic then $*$ is a pull back, and therefore $j = \widehat{d} \circ s = t_\Omega \circ R(d) \circ s = id_\Omega$.

3) $\Rightarrow$ 4): Consider the following diagram

$$\begin{array}{ccccccc}
RL(A) & \xrightarrow{RL(\{\ \}_A)} & RL(\Omega^A) & \xrightarrow{R(\widehat{L}_{A,\Omega})} & R(L(\Omega)^{L(A)}) & \xrightarrow{\widehat{R}_{L(A),L(\Omega)}} & RL(\Omega)^{RL(A)} \\
\uparrow{\scriptstyle t_A} & (i) & \uparrow{\scriptstyle t_{\Omega^A}} & & (ii) & & \downarrow{\scriptstyle RL(\Omega)^{t_A}} \\
A & \xrightarrow{\{\ \}_A} & \Omega^A & & \xrightarrow{{t_\Omega}^A} & & RL(\Omega)^A
\end{array}$$

(i) is commutative as t is a natural transformation, (ii) is commutative as t is a closed natural transformation, and as $(\ )^A$ preserves monomorphisms we have that ${t_\Omega}^A$ is monic if t_Ω is monic and consequently t_A is also a monomorphism.

This concludes the proof of Corollary 3.1. □

Corollary 3.2. *Let*

$$\underline{\mathbf{E}} \underset{R}{\overset{L}{\rightleftarrows}} \underline{\mathbf{E}}_0 , \qquad L \dashv R$$

be a geometric functor from $\underline{\mathbf{E}}_0$ to $\underline{\mathbf{E}}$, then the following statements are equivalent:

1) s is monic.

2) $\hat{d}$ is monic.

3) $L(s) \circ d = v_{\Omega_0}$.

4) v_{Ω_0} is a monomorphism.

5) v is a pointwise monic natural transformation.

Proof. As $s \circ \widehat{d} = t_{R(\Omega_0)} \circ R(L(s) \circ d)$ and as $\widehat{d} \dashv s$ we have that 1) $\iff$ 2) $\iff$ 3). Clearly 5) $\Rightarrow$ 4), and if v_{Ω_0} is monic then the following diagram which is commutative by the naturality of v

$$\begin{array}{ccc}
LR(\Omega_0) & \xrightarrow{v_{\Omega_0}} & \Omega_0 \\
\uparrow{\scriptstyle LR(true_0)} & & \uparrow{\scriptstyle true_0} \\
LR(1_0) & \underset{\sim}{\xrightarrow{v_{1_0}}} & 1_0
\end{array}$$

is a pull back as it is commutative, and therefore we have

$$v_{\Omega_0} = ch_{LR(\Omega_0)}(LR(true_0)) = L(s) \circ d.$$

This proves that 4) $\Rightarrow$ 3).

Finally, assume that $L(s) \circ d = v_{\Omega_0}$ and let $X \in |\underline{\mathbf{E}}_0|$, then

$$\delta_{LR(X)} = \tilde{L}_{R(X),R(X)} \circ L(\delta_{R(X)}) \circ d =$$

$\tilde{L}_{R(X),R(X)} \circ L(\tilde{R}_{X,X}) \circ LR(\delta_X) \circ L(s) \circ d =$

$\tilde{L}_{R(X),R(X)} \circ L(\tilde{R}_{X,X}) \circ LR(\delta_X) \circ v_{\Omega_0} =$

$\tilde{L}_{R(X),R(X)} \circ L(\tilde{R}_{X,X}) \circ v_{X\times X} \circ \delta_X = v_X \times v_X \circ \delta_X.$

This proves that v_x is a monomorphism and concludes the proof of Corollary 3.2. □

Lemma 3.1. *Let*

$$\underline{\mathbf{E}} \underset{R}{\overset{L}{\rightleftarrows}} \underline{\mathbf{E}}_0 , \qquad L \dashv R$$

be a geometric functor from $\underline{\mathbf{E}}_0$ to $\underline{\mathbf{E}}$,and let $X \in \underline{\mathbf{E}}_0$. Then the internal functor

$$\widehat{R}_{X,\Omega_0} : R({\Omega_0}^X) \longrightarrow R(\Omega_0)^{R(X)}$$

has a left and a right adjoint, $\overline{R}_X \dashv \widehat{R}_{X,\Omega_0} \dashv \ddot{R}_X$. Explicitly,

$\overline{R}_X = t_{R(\Omega_0)^{R(X)}} \circ R(\widehat{L}_{R(X),R(\Omega_0)} \circ {v_{\Omega_0}}^{LR(X)} \circ \exists_{v_X})$

$\ddot{R}_X = t_{R(\Omega_0)^{R(X)}} \circ R(\widehat{L}_{R(X),R(\Omega_0)} \circ {v_{\Omega_0}}^{LR(X)} \circ \forall_{v_X})$

The proof of Lemma 3.1 is a routine exercise which we leave to the reader.

Theorem 3.2. *Let*

$$\underline{\mathbf{E}} \underset{R}{\overset{L}{\rightleftarrows}} \underline{\mathbf{E}}_0 , \qquad L \dashv R$$

be a geometric functor from $\underline{\mathbf{E}}_0$ to $\underline{\mathbf{E}}$, then the direct image functor R preserves completeness of internally ordered objects.

Proof. Let $(H, \uparrow seg_H, inf_H) \in |\underline{C}(\underline{\mathbf{E}}_0)|$, then $(R(H), \uparrow seg_{R(H)})$ where

$$\uparrow seg_{R(H)} = R(H) \xrightarrow{R(\uparrow seg_H)} R({\Omega_0}^H) \xrightarrow{\widehat{R}_{H,\Omega_0}} R(\Omega_0)^{R(H)} \xrightarrow{s^{R(H)}} \Omega^{R(H)}$$

is an internally ordered object in $\underline{\mathbf{E}}$ as R is left exact.

By Theorem 3.1 we have that $\widehat{d} \dashv s$ and so $\widehat{d}^{R(H)} \dashv s^{R(H)}$.

By Lemma 3.1 we know that $\overline{R}_H \dashv \widehat{R}_{H,\Omega_0}$.

Finally, as $inf_H \perp \uparrow seg_H$, we get that $R(inf_H) \perp R(\uparrow seg_H)$ as R is left exact. It follows that

$$\widehat{d}^{R(H)} \circ \overline{R}_H \circ R(inf_H) \perp R(\uparrow seg_H) \circ \widehat{R}_{H,\Omega_0} \circ s^{R(H)} = \uparrow seg_{R(H)},$$

i.e. $(R(H), \uparrow seg_{R(H)}, inf_{R(H)}) \in |\underline{C}(\underline{\mathbf{E}})|$, where

$$inf_{R(H)} = \widehat{d}^{R(H)} \circ \overline{R}_H \circ R(inf_H) = t_{PR(H)} \circ R(\widehat{L}_{R(H),\Omega} \circ d^{LR(H)} \circ \exists_{v_H} \circ inf_H)$$

By virtually the same argument we get that

$$sup_{R(H)} = \widehat{d}^{R(H)} \circ \overline{R}_H \circ R(sup_H) = t_{PR(H)} \circ R(\widehat{L}_{R(H),\Omega} \circ d^{LR(H)} \circ \exists_{v_H} \circ sup_H)$$

□

Corollary 3.3. *The direct image functor of a geometric functor preserves complete Heyting algebra objects.*

Proof. The preservation of completeness follows from Theorem 3.2, and as the functor in question is left exact it preserves all finitary operations, in particular it preserves implication. □

Remark. If $F : \underline{\mathbf{E}} \longrightarrow \underline{\mathbf{E}}_0$ is a left exact functor on elementary topoi, then $(F(\Omega), \uparrow seg_{F(\Omega)})$ is a Heyting algebra object in $\underline{\mathbf{E}}_0$, and $\Rightarrow_{F(\Omega)} = \tilde{F}_{\Omega,\Omega} \circ F(\Rightarrow)$.

The statement that F preserves implication is of another nature. It means that the character $d = ch_{F(\Omega)}(F(true))$ preserves implication, i.e. on the level of subobjects this means that if $A, B \in P_*(C)$ then $F(A \Rightarrow B) = F(A) \Rightarrow F(B)$ in $P_*(C)$ is generally valid.

Theorem 3.3. *Let $F : \underline{\mathbf{E}} \longrightarrow \underline{\mathbf{E}}_0$ be a left exact functor on elementary topoi, then*

1) *F preserves implication iff $d = ch_{F(\Omega)}(F(true))$ is monic.*

2) *F preserves universal quantification iff F preserves implication and $\hat{F}$ is internally faithful (i.e. $\hat{F}$ is a pointwise monic natural transformation).*

Proof. Ad 1).

If F preserves implication, i.e. $\forall I \in |\underline{\mathbf{E}}_0| \quad \forall a, b \in F(\Omega) : (a \Rightarrow b) \circ d = a \circ d \Rightarrow b \circ d$, then d reflects the ordering and so d must be a monomorphism.

Conversely, if d is monic and $a \circ d \leqslant b \circ d$, then $a \leqslant b$ as d preserves binary intersection, whence $(a \Rightarrow b) \circ d = !_I \circ F^0 \circ F(true) \circ d = !_I \circ true_0$, i.e. F preserves implication.

Ad 2).

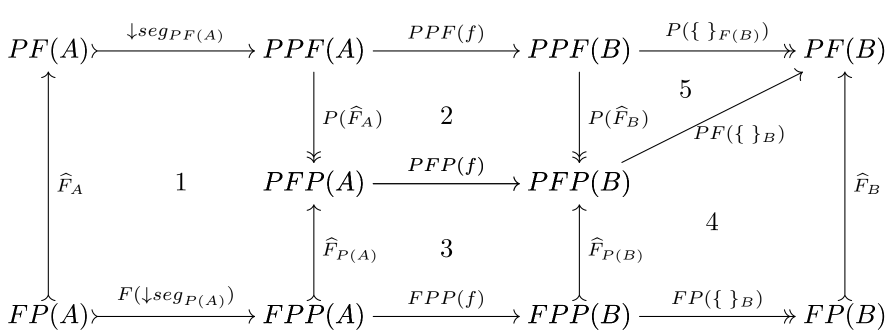
Observe that $\widehat{F}$ is pointwise monic iff $\forall A \in |\underline{\mathbf{E}}|$ the value $\widehat{F}_{A,\Omega} : F(\Omega^A) \longrightarrow F(\Omega)^{F(A)}$ is monic. This follows from the proof of Theorem 1.1.

If F preserves implication and is internally faithful, then

$$\widehat{F}_A : F(\Omega^A) \xrightarrow{\widehat{F}_{A,\Omega}} F(\Omega)^{F(A)} \xrightarrow{d^{F(A)}} {\Omega_0}^{F(A)}$$

defines a pointwise monic natural transformation.

Let $f \in \mathrm{Hom}_{\underline{\mathbf{E}}}(A, B)$. We claim that $F(\forall_f) \circ \widehat{F}_B = \widehat{F}_A \circ \forall_{F(f)}$, but this follows from commutativity of the diagram



1) is commutative as $\widehat{F}_A$ reflects order, being left exact and monic.

2), 3) and 4) are commutative by the naturality of $\widehat{F}$.

5) is commutative as F is left exact.

Conversely, assume that F preserves universal quantification, then F preserves implication. Indeed, notice that in

$$\begin{array}{ccccc}
\Omega & \xrightarrow{\langle true_\Omega, id_\Omega \rangle} & \Omega \times \Omega & \xrightarrow{\Rightarrow} & \Omega \\
{\scriptstyle true}\uparrow & 6 & \uparrow & P.B. & \uparrow{\scriptstyle true} \\
1 & \longrightarrow & \leqslant & \longrightarrow & 1
\end{array}$$

the front square, 6), is both a pull back and a universal quantification diagram, and therefore it follows that d preserves implication, - by chasing $true_0$ by the external Beck-condition in the following pull back diagram

$$\begin{array}{ccccc}
 & & F(\Omega) & \xrightarrow{\langle F(true_\Omega), id_{F(\Omega)} \rangle} & F(\Omega) \times F(\Omega) \\
 & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d \times d} \\
1_0 & \xrightarrow{true_0} & \Omega_0 & \xrightarrow{\langle true_{\Omega_0}, id_{\Omega_0} \rangle} & \Omega_0 \times \Omega_0
\end{array}$$

Finally, we notice that the diagram

$$\begin{array}{ccc}
FP(A) & \xrightarrow{\widehat{F}_A} & PF(A) \\
\downarrow{\scriptstyle F(\forall_A)} & & \downarrow{\scriptstyle \forall_{F(A)}} \\
F(\Omega) & \xrightarrow{d} & \Omega_0
\end{array}$$

is commutative, and therefore

$$\begin{array}{ccc}
FP(A) & \xrightarrow{\widehat{F}_A} & PF(A) \\
\uparrow{\scriptstyle F(\ulcorner true_A \urcorner)} & & \uparrow{\scriptstyle \ulcorner true_{F(A)} \urcorner} \\
F(1) & \xrightarrow[\sim]{} & 1_0
\end{array}$$

is a pull back. Now, by the above we know that d preserves implication, and as $\widehat{F}$ is natural we see that $\widehat{F}_A$ preserves implication. It follows that $\widehat{F}_A$ reflects the ordering, and so $\widehat{F}_A$ must be a monomorphism. Thus F is internally faithful. □

Corollary 3.4. *The direct image functor R of a geometric functor on elementary topoi preserves universal quantification iff it is full and faithful.*

Proof. Let

$$\underline{\mathbf{E}} \underset{R}{\overset{L}{\rightleftarrows}} \underline{\mathbf{E}}_0 \, , \qquad L \dashv R$$

be a geometric functor on elementary topoi. As L preserves binary cartesian products and the terminal object we have that $\widehat{R}$ is pointwise monic iff the end adjunction v is pointwise epic. Also, by Corollary 3.2 we have that s is monic iff v is pointwise monic. It follows that R preserves universal quantification iff v is a natural isomorphism iff R is full and faithful. □

Corollary 3.5. *The inverse image functor L of a geometric functor on elementary topoi preserves universal quantification iff*

the internal functor $\widehat{d} : \Omega \longrightarrow R(\Omega_0)$ has a left adjoint iff

$d : L(\Omega) \longrightarrow \Omega_0$ and $\widehat{L}_{\Omega,\Omega} : L(\Omega^\Omega) \longrightarrow L(\Omega)^{L(\Omega)}$ are monomorphisms.

Proof. Consider the following commutative diagram

$$\begin{array}{ccccc}
RL(\Omega^X) & \xrightarrow{R(\widehat{L}_{X,\Omega})} & R(L(\Omega)^{L(X)}) & \xrightarrow{R(d^{L(X)})} & R({\Omega_0}^{L(X)}) \\
 & & \downarrow{\scriptstyle \widehat{R}_{L(X),L(\Omega)}} & & \downarrow{\scriptstyle \widehat{R}_{L(X),\Omega_0}} \\
\uparrow{\scriptstyle t_{\Omega^X}} & & RL(\Omega)^{RL(X)} & \xrightarrow{R(d)^{RL(X)}} & R(\Omega_0)^{RL(X)} \\
 & & \downarrow{\scriptstyle RL(\Omega)^{t_X}} & & \downarrow{\scriptstyle R(\Omega_0)^{t_X}} \\
\Omega^X & \xrightarrow{(t_\Omega)^X} & RL(\Omega)^X & \xrightarrow{R(d)^X} & R(\Omega_0)^X
\end{array}$$

If $\widehat{d} = t_\Omega \circ R(d)$ has a left adjoint $a : R(\Omega_0) \longrightarrow \Omega$ then for all X in $\underline{\mathbf{E}}$ we have that

$$A_X = \widehat{R}_{L(X),\Omega_0} \circ R(\Omega_0)^{t_X} \circ a^X \dashv t_{\Omega^X} \circ R(\widehat{L}_{X,\Omega} \circ d^{L(X)}) = B_X.$$

Now if $f \in \mathrm{Hom}_{\underline{\mathbf{E}}}(X,Y)$ then $RPL(f) \circ A_X = A_Y \circ P(f)$. It follows that $t_{P(X)} \circ R(\widehat{L}_X \circ \forall_{L(f)}) = B_X \circ R(\forall_{L(f)}) = \forall_f \circ B_Y = t_{P(X)} \circ R(L(\forall_f) \circ \widehat{L}_Y)$, i.e.

$$L(\forall_f) \circ \widehat{L}_Y = \widehat{L}_X \circ \forall_{L(f)}.$$

This proves that if $\widehat{d}$ has a left adjoint then L preserves universal quantification.

Conversely, assume that d and $\widehat{L}_{\Omega,\Omega}$ are monic. We claim that $\widehat{d}$ has a left adjoint, i.e. that $\widehat{d}$ is inf-preserving. Whence we must prove that

$$inf_\Omega \circ \widehat{d} = t_{P(\Omega)} \circ RL(inf_\Omega) \circ R(d) = t_{P(\Omega)} \circ R(L(inf_\Omega) \circ d)$$

is equal to

$$\exists_{\widehat{d}} \circ inf_{R(\Omega_0)} = \exists_{\widehat{d}} \circ \widehat{d}^{R(\Omega_0)} \circ \overline{R}_{\Omega_0} \circ R(inf_{\Omega_0}) =$$

$$\exists_d \circ t_{PR(\Omega_0)} \circ R(\widehat{L}_{R(\Omega_0),\Omega} \circ d^{LR(\Omega_0)} \circ \exists_{v_{\Omega_0}} \circ inf_{\Omega_0}) =$$

$$t_{P(\Omega)} \circ R(\widehat{L}_\Omega \circ \exists_d \circ inf_{\Omega_0}),$$

i.e. we must prove that

$$L(inf_\Omega) \circ d \qquad \text{equals} \qquad \widehat{L}_\Omega \circ \exists_d \circ inf_{\Omega_0}$$

but this follows from noting that all the squares in the below diagrams are pull backs.

$$\begin{array}{ccccccc}
LP(\Omega) & \overset{\widehat{L}_\Omega}{\rightarrowtail} & PL(\Omega) & \xrightarrow{\exists_d} & P(\Omega_0) & \xrightarrow{inf_{\Omega_0}} & \Omega_0 \\
\uparrow{\scriptstyle L(\exists_{true})} & & \uparrow{\scriptstyle \exists_{L(true)}} & & \uparrow{\scriptstyle \exists_{true_0}} & & \uparrow{\scriptstyle true_0} \\
LP(1) & \underset{\widehat{L}_1}{\rightarrowtail} & PL(1) & \xrightarrow[\exists!_{L(1)}]{} & P(1_0) & \longrightarrow & 1_0
\end{array}$$

and

$$\begin{array}{ccccc}
LP(\Omega) & \xrightarrow{L(inf_\Omega)} & L(\Omega) & \xrightarrow{d} & \Omega_0 \\
\uparrow{\scriptstyle L(\exists_{true})} & & \uparrow{\scriptstyle L(true)} & & \uparrow{\scriptstyle true_0} \\
LP(1) & \xrightarrow[L(!_{P(1)})]{} & L(1) & \xrightarrow[!_{L(1)}]{} & 1_0
\end{array}$$

This concludes the proof of Corollary 3.5 □

Remark. Observe that the inverse image functor of a geometric functor on elementary Boolean topoi must necessarily preserve universal quantification as in this case $d : L(2) \longrightarrow 2_0$ is an isomorphism, and as the classical logic yields that

$$\forall_A : 2^A \xrightarrow{\neg^A} a^A \xrightarrow{\exists_A} 2 \xrightarrow{\neg} 2$$

for any elementary Boolean topos.

We refer the reader to [9] for a nontrivial application of Theorem 3.3.

We finish this chapter with an application of Stone's theorem for elementary topoi to the theory of logical functors.

Theorem 3.4. *Let $L : \underline{\mathbf{E}} \longrightarrow \underline{\mathbf{E}}_0$ be a logical functor on elementary topoi, then L has a left adjoint $T : \underline{\mathbf{E}}_0 \longrightarrow \underline{\mathbf{E}}$ iff L has a right adjoint $R : \underline{\mathbf{E}}_0 \longrightarrow \underline{\mathbf{E}}$.*

Proof. Consider the following diagram

$$\begin{array}{ccc}
\underline{\mathbf{E}}^{op} & \xrightarrow{L^{op}} & \underline{\mathbf{E}}_0^{op} \\
\downarrow{\scriptstyle P} & & \downarrow{\scriptstyle P} \\
\underline{\mathbf{E}} & \xrightarrow{L} & \underline{\mathbf{E}}_0
\end{array}$$

which is commutative up to natural isomorphism as L is a logical functor. In Chapter 2 we saw that the vertical functors are tripleable, and so it follows from the general theory of monads that a left adjoint T of L gives rise to a a left adjoint L^{op} as $\underline{\mathbf{E}}^{op}$ has coequalizers. But this means that L has a right adjoint. Thus, the existence of T implies the existence of R.

Suppose for a moment that we have a local homeomorphism. Then the transformation θ yields a system of natural isomorphisms

$$\theta_X = \theta_{X,\Omega} : PT(X) \longrightarrow R(L(\Omega)^X)$$

This means that $R(L(\Omega)^X)$ is orderisomorphic to $PT(X)$ and $T(X)$ can therefore be described as the singletons in $R(L(\Omega)^X)$.

Now assume that L has a right adjoint R. From Corollary 3.1, Proposition 2.13 and the fact that L is logical we get that if $X \in |\underline{\mathbf{E}}_0|$ then $R(L(\Omega)^X)$ is a complete Heyting algebra in $\underline{\mathbf{E}}$, and if $f \in \mathrm{Hom}_{\underline{\mathbf{E}}_0}(X,Y)$ then $R(L(\Omega)^f)$ generates a Stone morphism

$$(E_f, R(L(\Omega)^f), A_f) : R(L(\Omega)^X) \longrightarrow R(L(\Omega)^Y).$$

Thus if we define the functor $T : \underline{\mathbf{E}}_0 \longrightarrow \underline{\mathbf{E}}$ by letting

$$T(X) \xrightarrow{\;i_X\;} R(L(\Omega)^X)$$

be the extension of the atoms in $R(L(\Omega)^X)$, and $T(f)$ the restriction of E_f to the extension of the atoms, i.e. such that

$$\begin{array}{ccc} R(L(\Omega)^X) & \xrightarrow{\;E_f\;} & R(L(\Omega)^Y) \\ \uparrow{\scriptstyle i_X} & & \uparrow{\scriptstyle i_Y} \\ T(X) & \xrightarrow{\;T(f)\;} & T(Y) \end{array}$$

is commutative. Furthermore we have the end adjunction from Proposition 2.17, i.e. the Stone morphism

$$PT(X) \;\underset{f_*}{\overset{f_!}{\rightrightarrows}}\; \overset{f^*}{\twoheadleftarrow} \; R(L\Omega)^X$$

where

$$f_! = \exists_{i_X} \circ sup_{R(L(\Omega)^X)}$$

$$f^* = \downarrow seg_{R(L(\Omega)^X)} \circ P(i_X) \qquad \text{and}$$

$f_! \dashv f^* \dashv f_*$ and f^* preserving implication and $id_{PT(X)} = f_! \circ f^*$.

Sublemma 3.1. *The front and end adjunction t and v for $L \dashv R$ (with L logical) have left and right adjoints on all complete lattices in $\underline{\mathbf{E}}$ and $\underline{\mathbf{E}}_0$. In particular they generate Stone morphisms whenever possible.*

Proof. Let A be a complete lattice in $\underline{\mathbf{E}}$ and consider the following diagram.

$$\begin{array}{ccccccc} PRL(A) & \xrightarrow{\widehat{d}^{RL(A)} \circ \overline{R}_{L(A)}} & RPL(A) & \xrightarrow{R(\widehat{L}_A^{-1})} & RLP(A) & \xrightarrow{RL(sup_A/inf_A)} & RL(A) \\ \uparrow{\scriptstyle \exists_{t_A}} & & (1) & & \uparrow{\scriptstyle t_{P(A)}} & (2) & \uparrow{\scriptstyle t_A} \\ P(A) & & \xrightarrow{\quad id_{P(A)} \quad} & & P(A) & \xrightarrow{sup_A/inf_A} & A \end{array}$$

As L preserves existential quantification in $\underline{\mathbf{E}}$ we know that

$$\exists_{t_A} \circ \widehat{d}^{RL(A)} \circ \overline{R}_{L(A)} = t_{P(A)} \circ R(\hat{L}_A).$$

It follows that (1) is commutative. The square (2) is commutative by the naturality of t. Thus we see that t_A is both sup- and inf-preserving. Furthermore, as t is natural it follows that t_A

must preserve all finitary operations on A. In particular if A is a complete Heyting algebra in $\underline{\mathbf{E}}$, then t_A generates a Stone morphism

$$RL(A) \underset{t_{A*}}{\overset{t_A!}{\rightrightarrows}} \xleftarrow{\ t_A{}^*\ } A$$

If Y is a complete lattice in $\underline{\mathbf{E}}_0$ we consider the diagram

$$\begin{array}{ccccccc}
PLR(Y) & \xrightarrow{\widehat{L}^{-1}_{R(Y)}} & LPR(Y) & \xrightarrow{L(\widehat{d}^{R(Y)}\circ\overline{R}_Y)} & LRP(Y) & \xrightarrow{LR(sup_Y/inf_Y)} & LR(Y) \\
\downarrow \exists_{v_Y} & & (3) & & \downarrow v_{P(Y)} & (4) & \downarrow v_Y \\
P(Y) & & \xrightarrow{id_{P(Y)}} & & P(Y) & \xrightarrow{sup_Y/inf_Y} & Y
\end{array}$$

The square (3) is commutative as

$$L(\widehat{d}^{R(Y)} \circ \overline{R}_Y) \circ v_{P(Y)} = \widehat{L}_{R(Y)} \circ \exists_{v_Y}$$

is simply the exponential adjoint of

$$\widehat{d}^{R(Y)} \circ \overline{R}_Y = t_{PR(Y)} \circ R(\widehat{L}_{R(Y)} \circ \exists_{v_Y}),$$

and we noted in the proof of Theorem 3.2 that this equation is valid for any geometric functor.

Thus we see that v_Y is both sup- and inf-preserving. Again, if Y is a complete Heyting algebra in $\underline{\mathbf{E}}_0$, then v_Y generates a Stone morphism

$$Y \underset{v_{Y*}}{\overset{v_Y!}{\rightrightarrows}} \xleftarrow{\ v_Y{}^*\ } LR(Y)$$

This concludes the proof of Sublemma 3.1. □

We can now construct the front and end adjunction for the asserted situation $T \dashv L$. Consider for $X \in \underline{\mathbf{E}}_0$ the following diagram

$$\begin{array}{ccccc}
P(X) & \xrightarrow{(d^{-1})^X} & L(\Omega)^X & \xrightarrow{v_{L(\Omega)^X}!} & LR(L(\Omega)^X) \\
\uparrow \{\ \}_X & & & & \uparrow L(i_X) \\
X & & \xrightarrow{\eta_X} & & LT(X)
\end{array}$$

In this diagram the upper morphism is the outer left adjoint of a Stone morphism, and as L is logical the two vertical morphisms are extensions of atoms. Thus η_X exists and is determined by the commutativity of the diagram. Furthermore

$$\eta = \{\eta_X : X \longrightarrow LT(X)\}_{X \in |\underline{\mathbf{E}}_0|}$$

is a natural transformation, $\eta : id_{\underline{\mathbf{E}}_0} \Rightarrow LT$.

The end adjunction $\epsilon : TL \Rightarrow id_{\underline{\mathbf{E}}}$ is defined for $A \in |\underline{\mathbf{E}}|$ by the following commutative diagram

$$\begin{array}{ccccc}
R(L(\Omega)^{L(A)}) & \xrightarrow{R(\widehat{L}^{-1}_{A,\Omega})} & RLP(A) & \xrightarrow{t_{P(A)}!} & P(A) \\
\big\uparrow{\scriptstyle i_{L(A)}} & & & & \big\uparrow{\scriptstyle \{\ \}_A} \\
TL(A) & & \xrightarrow{\qquad\epsilon_A\qquad} & & A
\end{array}$$

whose existence follows as the upper morphism is the outer left adjoint of a Stone morphism.

<u>$\eta_{L(A)} \circ L(\epsilon_A) = id_{L(A)}$</u>

Consider

$$\begin{array}{ccccc}
PL(A) & \xrightarrow{(d^{-1})^{L(A)} \circ v_{L(\Omega)^{L(A)}}!} & LR(L(\Omega)^{L(A)}) & \xrightarrow{L(R(\widehat{L}^{-1}_{A}) \circ t_{P(A)}!)} & LP(A) \\
\big\uparrow{\scriptstyle \{\ \}_{L(A)}} & & \big\uparrow{\scriptstyle L(i_{L(A)})} & & \big\uparrow{\scriptstyle L(\{\ \}_A)} \\
L(A) & \xrightarrow{\eta_{L(A)}} & LTL(A) & \xrightarrow{L(\epsilon_A)} & L(A)
\end{array}$$

The right adjoint of the upper morphism is

$$L(t_{P(A)}) \circ LR(\widehat{L}_{A,\Omega}) \circ v_{L(\Omega)^{L(A)}} \circ d^{L(A)} = \widehat{L}_{A,\Omega} \circ d^{L(A)} = \widehat{L}_A$$

which is an isomorphism, and as $L(\{\ \}_A) \circ \widehat{L}_A = \{\ \}_{L(A)}$ it follows that $\eta_{L(A)} \circ L(\epsilon_A) = id_{L(A)}$.

<u>$R(L(\Omega)^X)$ is atomic.</u>

We shall verify this statement by showing that

$$f^* = \downarrow seg_{R(L(\Omega)^X)} \circ P(i_X)$$

is a monomorphism.

We do this by verifying that

$$t_{PT(X)} \circ R(\widehat{L}_{T(X),\Omega}) \circ R(L(\Omega)^{\eta_X})$$

is a right inverse (and hence the inverse) of f^*.

Indeed, consider the diagram on the following page. (1),(2), (3) and (5) are commutative by naturality. (4) commutes by the definition of $\widehat{L}$. (6) commutes as L is left exact. (7) commutes by construction of η_X. (8) is commutative as both morphisms in the square are internal functors which are right adjoint of $(d^{-1})^X \circ v_{L(\Omega)^X}!$.

Finally, using that $R(d^X)$ is monic we get that

$$\downarrow seg_{R(L(\Omega)^X)} \circ P(i_X) \circ t_{PT(X)} \circ R(\widehat{L}_{T(X),\Omega}) \circ R(L(\Omega)^{\eta_X}) = id_{R(L(\Omega)^X)}.$$

This proves that $R(L(\Omega)^X)$ is atomic.

$\underline{T(\eta_X) \circ L(\epsilon_{T(X)}) = id_{T(X)}}$

Consider

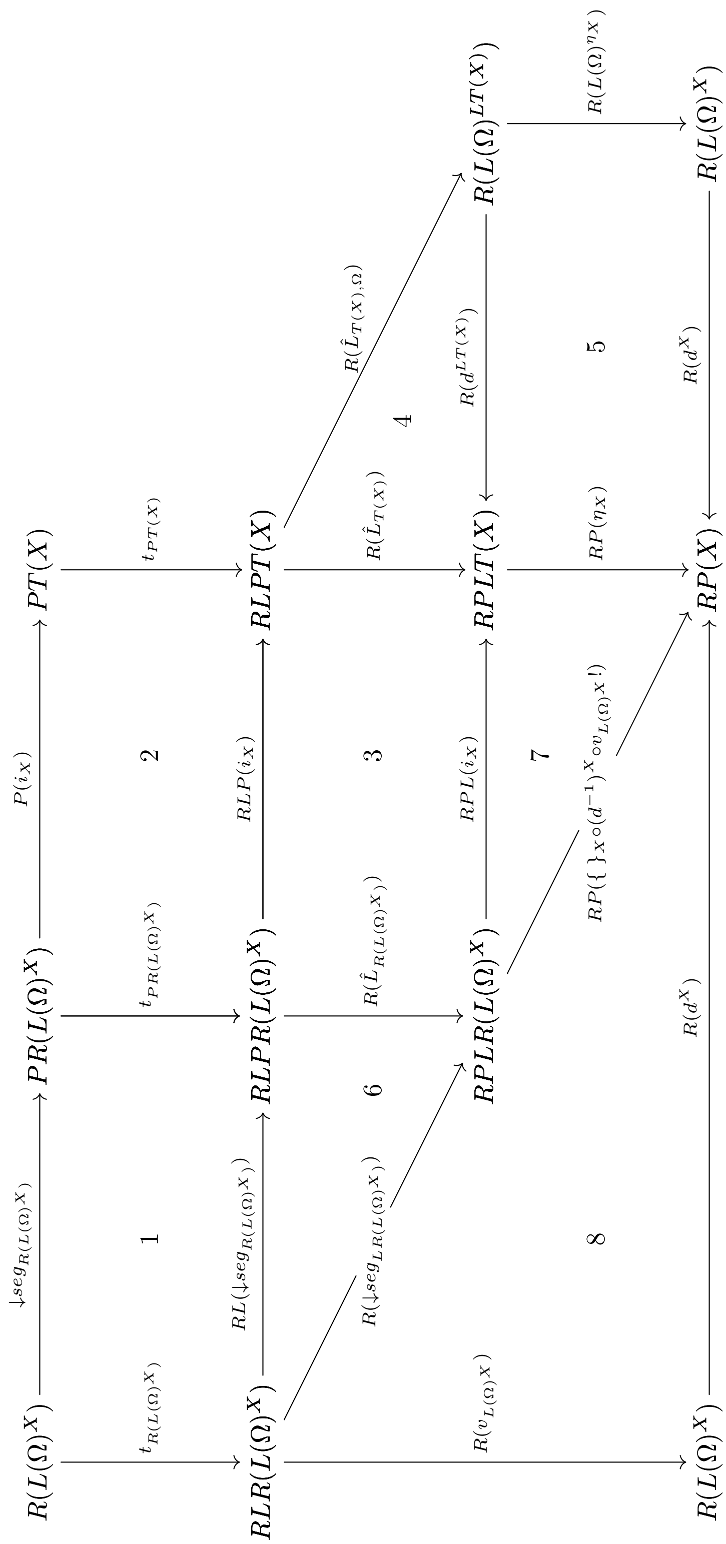
$$\begin{array}{ccccc} R(L(\Omega)^X) & \xrightarrow{E_{\eta_X}} & R(L(\Omega)^{LT(X)}) & \xrightarrow{R(\widehat{L}^{-1}_{T(X),\Omega}) \circ t_{PT(X)!}} & PT(X) \\ \uparrow{\scriptstyle i_X} & & \uparrow{\scriptstyle i_{LT(X)}} & & \uparrow{\scriptstyle \{\ \}_{T(X)}} \\ T(X) & \xrightarrow{T(\eta_X)} & TLT(X) & \xrightarrow{\epsilon_{T(X)}} & T(X) \end{array}$$

The right adjoint of the upper morphism is

$$t_{PT(X)} \circ R(\widehat{L}_{T(X),\Omega}) \circ R(L(\Omega)^{\eta_X}).$$

Now, we have just seen that this morphism is the inverse of f^*, and as $\{\ \}_{T(X)} = i_X \circ f^*$, it follows that $T(\eta_X) \circ L(\epsilon_{T(X)}) = id_{T(X)}$.

This concludes the proof of Theorem 3.4. □


$R(L(\Omega)^X)$
$PR(L(\Omega)^X)$
$PT(X)$
$RLR(L(\Omega)^X)$
$RLPR(L(\Omega)^X)$
$RLPT(X)$
$RPLR(L(\Omega)^X)$
$RPLT(X)$
$R(L(\Omega)^{LT(X)})$
$R(L(\Omega)^X)$
$RP(X)$
$R(L(\Omega)^X)$
$\downarrow seg_{R(L(\Omega)^X)}$
$P(i_X)$
$t_{R(L(\Omega)^X)}$
$t_{PR(L(\Omega)^X)}$
$t_{PT(X)}$
$RL(\downarrow seg_{R(L(\Omega)^X)})$
$RLP(i_X)$
$R(\downarrow seg_{LR(L(\Omega)^X)})$
$R(\hat{L}_{R(L(\Omega)^X)})$
$R(\hat{L}_{T(X)})$
$R(\hat{L}_{T(X),\Omega})$
$RPL(i_X)$
$R(d^{LT(X)})$
$R(v_{L(\Omega)^X})$
$RP(\{\ \}_X \circ (d^{-1})^X \circ v_{L(\Omega)^X}!)$
$RP(\eta_X)$
$R(L(\Omega)^{\eta_X})$
$R(d^X)$
$R(d^X)$
1
2
3
4
5
6
7
8

In order to prevent a few wild conjectures let us mention one or two examples showing what is not the case.

Example 1. Let

$$\mathbb{C} = \{\ \overset{g\circ f}{\circlearrowright}\ \underset{id_1}{1} \underset{f}{\overset{g}{\rightleftarrows}} \underset{id_0}{0}\ \} \quad \text{and} \quad \mathbb{B} = \{\ \underset{id_1}{1} \xleftarrow{\ f\ } \underset{id_0}{0}\ \}$$

and let

$$L : \underline{Sets}^{\mathbb{C}^{op}} \longrightarrow \underline{Sets}^{\mathbb{B}^{op}}$$

be the functor induced by the inclusion $\mathbb{B} \hookrightarrow \mathbb{C}$, then L is faithful and preserves the subobject classifier but not universal quantification. L has a left and a right adjoint, but L is not a logical functor.

Example 2. Let $\mathbb{G}$ be a topological group and let

$$I : \underline{B}_{\mathbb{G}} \hookrightarrow \underline{B}_G$$

be the full subcategory of discrete continuous representations of G (the underlying group of $\mathbb{G}$) in Sets. I. e. if $(X, \cdot) \in |\underline{B}_G|$ then $(X, \cdot) \in |\underline{B}_{\mathbb{G}}|$ iff $\forall x \in X$ the isotropic group

$$G_x = \{g \in G | x \cdot g = x\}$$

is an open subgroup of G. $\underline{B}_{\mathbb{G}}$ is a coreflective subcategory of $\underline{B}_G$ and the coreflector is the direct image functor of a geometric functor on elementary Boolean topoi. The inclusion functor I preserves the subobject classifier and universal quantification. It is logic iff it has a left adjoint iff there is a smallest open set containing the unit.

If $f : \mathbb{G} \longrightarrow \mathbb{H}$ is a map of topological groups, then the induced geometric functor

$$\underline{B}_{\mathbb{G}} \underset{f_*}{\overset{f^*}{\rightleftarrows}} \underline{B}_{\mathbb{H}}$$

is an essential functor iff the the image in H of any open subgroup of G is contained in a smallest open subgroup of H. In case $\mathbb{G}$ and $\mathbb{H}$ are Boolean groups, i. e. if the topologies are totally disconnected compact Hausdorffian, then the above condition is equivalent to either f preserves open subgroups or $f(G)$ is an open subgroup of H or the index $[f(G) : H]$ is finite.

Example 3. Let $\mathbb{C}$ be the small category from example 1, and consider

$$\underline{Sets}^{\mathbb{C}^{op}} \underset{\Delta}{\overset{\Gamma}{\leftrightarrows}} \underline{Sets}$$

where $\Delta \dashv \Gamma$. In this particular case Δ is both left <u>and</u> right adjoint to Γ. The topology generated by $\Delta \dashv \Gamma$ is the double negation. Thus we have an example of a "finite" topos defined over <u>Sets</u> such that its centre exists and such that the centre is not closed.

This answers in the negative the first question of [1] (SLN 270) Exposé VI 8.4.7.

Appendix A

Transfinite Induction in Elementary Topoi

In this appendix we shall study the classical concept of transfinite induction in the context of elementary topoi. Officially we establish the recursion theorem, but we believe that the importance of this work lies in the method of proof rather than in the results themselves. In fact the question we shall investigate is that of constructing a morphism $f : A \longrightarrow X$ in an elementary topos $\underline{\mathbf{E}}$, such that f is the solution of some problem Q. We may think of f as being described by its graph $\Gamma_f : A \rightarrowtail A \times X$ which means that what we are looking for is a relation R from A to X. We know, however, that such constructions can be effected by the fixpoint theorem, i.e. we must study Q in order to obtain the idea of an approximative relational solution. Having done this we take the internal intersection of all such, and by the fixpoint theorem we therefore have a smallest relational solution $R \rightarrowtail A \times X$. In order to verify that R is a graph we start by checking if $R \longrightarrow A$ is a monomorphism, i.e. if R is a partial graph. Once more we find that the elementary topos supplies the method: Consider the partial graph generated by R. This is done by taking the unique existentiation of $R \longrightarrow A$ and restricting R to this subobject. Thus we only need to check that this subrelation is a fixpoint for the internal functor defining R to get that the two relations agree, i.e. R is a partial graph. Finally to see that the relation is globally defined we must once more study the setting and the universal property described in Q to see that this is actually the case.

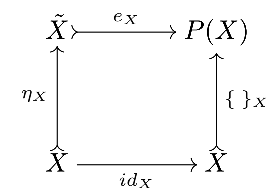
The recursion theorem established below can now be seen as a mere example of this general method of constructing a morphism.

Let $\underline{\mathbf{E}}$ be a fixed elementary topos. We shal use the partial graph operator in $\underline{\mathbf{E}}$ and some of its fundamental properties which we record below for reference.

Recall the construction of $\sim$ in $\underline{\mathbf{E}}$. Let

1) $$\tilde{X} \overset{e_X}{\rightarrowtail} P(X) \underset{id_{P(X)}}{\overset{\{\ \}_{P(X)} \circ P(\{\ \}_X)}{\rightrightarrows}} P(X)$$

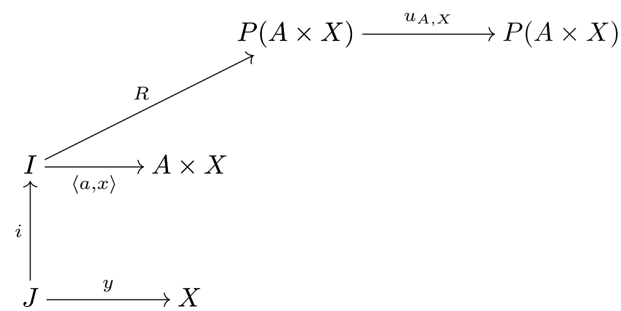
be an equalizer diagram. Notice that $\{\ \}_{P(X)} \circ P(\{\ \}_X) \leqslant id_{P(X)}$ and as $\{\ \}_{P(X)} \circ P(\{\ \}_X)$ is idempotent, it follows that the equalizer 1) is preserved by any functor. As $\{\ \}_X \circ \{\ \}_{P(X)} \circ P(\{\ \}_X) = \{\ \}_X$ we get the following commutative square (and pull back):



Applying $(\ )^A$ to 1) we get (up to isomorphism) the idempotent equalizer diagram

2) $$\tilde{X}^A \xrightarrow{g_{A,X}} P(A\times X) \underset{id_{P(A\times X)}}{\overset{u_{A,X}}{\rightrightarrows}} P(A\times X)$$

where $u_{A,X}$ is characterized by the following elementary description:



$\forall I \in |\underline{\mathbf{E}}| \quad \forall R \in P(A\times X) \quad \forall a \in A \quad \forall x \in X : \langle a, x\rangle \underline{\in} R \circ u_{A,X}$ iff

$\langle a, x\rangle \underline{\in} R$ and $\forall J \in |\underline{\mathbf{E}}| \quad \forall i \in I \quad \forall y \in X : \langle i \circ a, y\rangle \underline{\in} i \circ R$ implies that $y = i \circ x$.

P1. $g_{A,X} \circ \exists_{p_0} = (dom_X)^A : \tilde{X}^A \longrightarrow P(A)$, where $dom_X = ch_{\tilde{X}}(\eta_X) : \tilde{X} \longrightarrow \Omega$

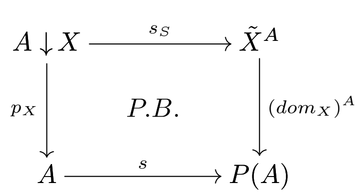
P2. If $R : I \longrightarrow P(A\times X)$ then R is a graph (i.e. R factors through $\Gamma_{A,X} : X^A \rightarrowtail P(A\times X)$) iff R is a partial graph (i. e. $R \circ u_{A,N} = R$) and R is globally defined (i.e. $R \circ \exists_{p_0} = !_I \circ \ulcorner true_A \urcorner$)

P3. If $R, S \in \mathrm{Hom}_{\underline{\mathbf{E}}}(I, P(A\times X))$ and $S \leqslant R \circ u_{A,X}$ then $S = S \circ u_{A,X}$,

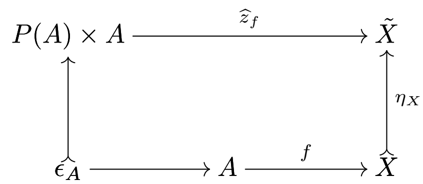
i.e. $u_{A,X} \circ \downarrow seg_{P(A\times X)} \circ \exists_{u_{A,X}} = u_{A,X} \circ \downarrow seg_{P(A\times X)}$.

P4. If $R, S, T \in \mathrm{Hom}_{\underline{\mathbf{E}}}(I, P(A\times X))$ and $R \leqslant T \circ u_{A,X}$, $S \leqslant T$ and $R \circ \exists_{p_0} = S \circ \exists_{p_0}$ then $R = S$.

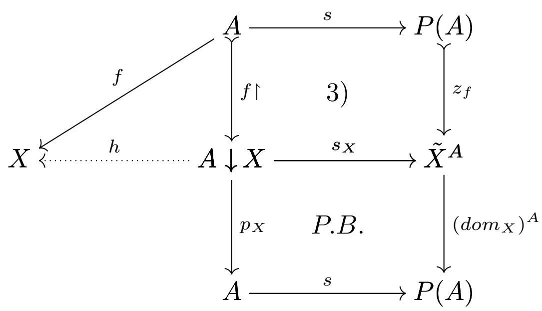
Let $s : A \longrightarrow P(A)$ be a relation in $\underline{\mathbf{E}}$. We shall think of s as a strict $\downarrow$-segment, but we shall not yet make any assumptions on s. If X is any object in $\underline{\mathbf{E}}$ we consider the representor $A \downarrow X$ of partial morphisms from A to X defined on the initial segment of s.



If $f \in \mathrm{Hom}_{\underline{\mathbf{E}}}(A, X)$ consider the lifting $\widehat{z}_f : P(A) \times A \longrightarrow \tilde{X}$ of f constructed out of the universal property of $\tilde{X}$, making the diagram



into a pull back. As $\widehat{z}_f \circ dom_X = ev_{A,\Omega}$ we have that $z_f \circ (dom_X)^A = id_{P(A)}$.



where $f\!\restriction \circ\, s_X = s \circ z_f$ and $f\!\restriction \circ\, p_X = id_A$. Observe that the square 3) is a pull back.

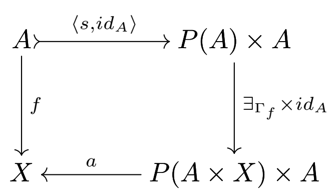
Definition A.1. *The relation $s : A \longrightarrow P(A)$ is said to satisfy* **the principle of transfinite induction** *iff* $\forall X \in |\underline{\mathbf{E}}| \quad \forall h : A \downarrow X \longrightarrow X \quad \exists!\ f : A \longrightarrow X$ *such that* $f_{\restriction} \circ h = f$.

If $f \in \mathrm{Hom}_{\underline{\mathbf{E}}}(A, X)$ we readily see that

4) $z_f \circ g_{A,X} = \exists_{\Gamma_f} : P(A) \rightarrowtail P(A \times X)$

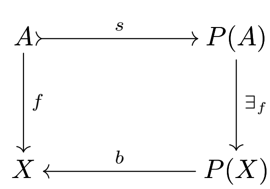
This leads to the following special cases of the principle of transfinite induction.

T1. $\forall X \in |\underline{\mathbf{E}}| \quad \forall a : P(A \times X) \times A \longrightarrow X \quad \exists!\ f : A \longrightarrow X$ such that



is commutative.

T2. $\forall X \in |\underline{\mathbf{E}}| \quad \forall b : P(X) \longrightarrow X \quad \exists!\ f : A \longrightarrow X$ such that



is commutative.

Indeed, if s satisfies the principle of transfinite induction we let $h \in \mathrm{Hom}_{\underline{\mathbf{E}}}(A \downarrow X, X)$ be the morphism $\langle s_X \circ g_{A,X}, p_X \rangle \circ a$ then $f_{\restriction} \circ h = f_{\restriction} \circ \langle s_X \circ g_{A,X}, p_X \rangle \circ a = \langle s \circ \exists_{\Gamma_f}, id_A \rangle \circ a$, i.e. T1 is satisfied.

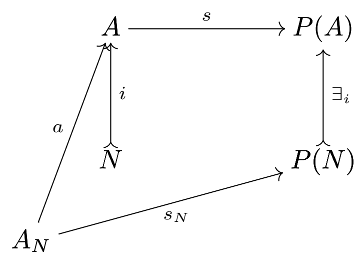
If T1 is valid we get T2 by letting $a = p_0 \circ \exists_{p_1} \circ b$.

What can be said on the uniqueness of the solution of a recursion problem?

Suppose $f, g \in \mathrm{Hom}_{\underline{\mathbf{E}}}(A, X)$ and $h \in \mathrm{Hom}_{\underline{\mathbf{E}}}(A \downarrow X, X)$ such that $f = f_{\upharpoonright} \circ h$ and $g = g_{\upharpoonright} \circ h$. Let

$$N \overset{i}{\rightarrowtail} A \underset{g}{\overset{f}{\rightrightarrows}} X$$

be the equalizer diagram for f and g, and consider the following pull back diagram:



We claim that $A_N \leqslant N$, i.e. $a \circ f = a \circ g$.

i) $a \circ f_{\upharpoonright} \circ p_X = a = a \circ g_{\upharpoonright} \circ p_X$.

ii) $a \circ f_{\upharpoonright} \circ s_X \circ g_{A,X} = a \circ s \circ z_f \circ g_{A,X} = a \circ s \circ \exists_{\Gamma_f} = s_N \circ \exists_i \circ \exists_{\Gamma_f} =$

$s_N \circ \exists_i \circ \exists_{\Gamma_g} = a \circ s \circ \exists_{\Gamma_g} = a \circ s \circ z_g \circ g_{A,X} = a \circ g_{\upharpoonright} \circ s_X \circ g_{A,X}$

and as $g_{A,X}$ is monic it follows that

$a \circ f_{\upharpoonright} \circ s_X = a \circ g_{\upharpoonright} \circ s_X$

iii) From i) and ii) we get that $a \circ f_{\upharpoonright} = a \circ g_{\upharpoonright}$, and therefore

$a \circ f = a \circ f_{\upharpoonright} \circ h = a \circ g_{\upharpoonright} \circ h = a \circ g$

Observe that stated internally the inequality $A_N \leqslant N$ reads

5) $\ulcorner ch_A(i) \urcorner \circ \downarrow seg_{P(A)} \circ P(s) \leqslant \ulcorner ch_A(i) \urcorner$.

Definition A.2. *The relation $s : A \longrightarrow P(A)$ is said to be* **inductive** *iff the internal functor $\downarrow seg_{P(A)} \circ P(s)$ on $P(A)$ has a unique fixpoint (namely $\ulcorner true_A \urcorner$) defined over 1.*

Remark. From the proof of the fixpoint theorem we know that the inequality 5) gives rise to a subobject $j : N \rightarrowtail A$ such that $\ulcorner ch_A(j) \urcorner \circ \downarrow seg_{P(A)} \circ P(s) = \ulcorner ch_A(j) \urcorner$ and such that $\ulcorner ch_A(j) \urcorner \leqslant \ulcorner ch_A(i) \urcorner$. Thus we conclude that inductive relations satisfy the uniqueness property of the transfinite induction property.

Conversely, assume that $s \in \mathrm{Hom}_{\underline{\mathbf{E}}}(A, P(A))$ satisfies the uniqueness part of T2, and let $m \in \mathrm{Hom}_{\underline{\mathbf{E}}}(A, \Omega)$ such that $\ulcorner m \urcorner = \ulcorner m \urcorner \circ \downarrow seg_{P(A)} \circ P(s)$. From this it follows that $\ulcorner m \urcorner = \ulcorner m \urcorner \circ \downarrow seg_{P(A)} \circ P(s) = \ulcorner id_\Omega \urcorner \circ P(m) \circ \downarrow seg_{P(A)} \circ P(s) = \ulcorner id_\Omega \urcorner \circ \downarrow seg_{P(\Omega)} \circ P(\exists_m) \circ P(s) = \ulcorner ch_\Omega(true) \urcorner \circ \downarrow seg_{P(\Omega)} \circ P(s \circ \exists_m) = \ulcorner ch_{P(\Omega)}(\exists_{true}) \urcorner \circ P(s \circ \exists_m) = \ulcorner s \circ \exists_m \circ ch_{P(\Omega)}(\exists_{true}) \urcorner$, i.e. $m = s \circ \exists_m \circ ch_{P(\Omega)}(\exists_{true})$. This proves that s is inductive.

Theorem A.1. *Let $s : A \longrightarrow P(A)$ be an inductive relation in an elementary topos $\underline{\mathbf{E}}$, then s satisfies the principle of transfinite induction*

Proof. Let $X \in |\underline{\mathbf{E}}|$ and $h \in \mathrm{Hom}_{\underline{\mathbf{E}}}(A \downarrow X, X)$ be given. We have seen that there is at most one $f \in \mathrm{Hom}_{\underline{\mathbf{E}}}(A, X)$ such that $f_{\upharpoonright} \circ h = f$. We shall now give the explicit construction of this f.

Consider the internal functor

$$F = P(A \times X) \xrightarrow{\downarrow seg_{P(A\times X)}} PP(A \times X) \xrightarrow{P(s_X \circ g_{A,X})} P(A \downarrow X) \xrightarrow{\exists_{\langle p_X, h\rangle}} P(A \times X)$$

which to a relation R from A to X assigns to any partial morphism t from A to X whose domain is an initial segment $(a)s$ of s and whose graph is contained in R the value $(a, (t)h)$.

Explicitly,

$$\begin{array}{ccccccc}
 & J & \xrightarrow{t} & A \downarrow X & \xrightarrow{s_X} & \tilde{X}^A & \xrightarrow{g_{A,X}} P(A \times X) \\
 & {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle \langle p_X, h\rangle} & & & \\
 & I & \xrightarrow[\langle a,x\rangle]{} & A \times X & & & \\
 & {\scriptstyle R}\downarrow & & & & & \\
 P(A\times X) & & \xrightarrow{F} & & P(A \times X) & &
\end{array}$$

$\forall I \in |\underline{\mathbf{E}}| \quad \forall R \in P(A \times X) \quad \forall a \in A \quad \forall x \in X : \langle a, x\rangle \underline{\in} R \circ F$ iff

$\exists J \in |\underline{\mathbf{E}}|$, $\exists e \in I$ (epi), $\exists t \in A \downarrow X$ such that

$t \circ p_X = e \circ a$, $t \circ h = e \circ x$ and $t \circ s_X \circ g_{A,X} \leqslant e \circ R$.

By the fixpoint theorem we know that F has a smallest such fixpoint (defined over 1) $B : 1 \longrightarrow P(A \times X)$. We claim that B resolves our problem, i.e. that there exists $f \in \mathrm{Hom}_{\underline{\mathbf{E}}}(A, X)$ such that $B = \ulcorner f \urcorner \circ \Gamma_{A,X}$ and such that $f_{\upharpoonright} \circ h = f$.

Suppose that $f \in \mathrm{Hom}_{\underline{\mathbf{E}}}(A, X)$. Let us compute $\ulcorner f \urcorner \circ \Gamma_{A,X} \circ F$.

$\ulcorner f \urcorner \circ \Gamma_{A,X} \circ F = \ulcorner f \urcorner \circ \Gamma_{A,X} \circ \downarrow seg_{P(A\times X)} \circ P(g_{A,X}) \circ P(s_X) \circ \exists_{\langle p_X, h\rangle} =$

$\ulcorner true_A \urcorner \circ \exists_{\Gamma_f} \circ \downarrow seg_{P(A\times X)} \circ P(g_{A,X}) \circ P(s_X) \circ \exists_{\langle p_X, h\rangle} =$ (i)

$\ulcorner true_A \urcorner \circ \downarrow seg_{P(A)} \circ \exists_{\exists_{\Gamma_f}} \circ P(g_{A,X}) \circ P(s_X) \circ \exists_{\langle p_X, h\rangle} =$ (ii)

$\ulcorner true_A \urcorner \circ \downarrow seg_{P(A)} \circ \exists_{z_f} \circ P(s_X) \circ \exists_{\langle p_X, h\rangle} =$ (iii)

$\ulcorner true_A \urcorner \circ \downarrow seg_{P(A)} \circ P(s) \circ \exists_{f_{\upharpoonright}} \circ \exists_{\langle p_X, h\rangle} = \ulcorner true_A \urcorner \circ \exists_{f_{\upharpoonright}} \circ \exists_{\langle p_X, h\rangle} =$

$\ulcorner true_A \urcorner \circ \exists_{\Gamma_{f_{\upharpoonright} \circ h}} = \ulcorner f_{\upharpoonright} \circ h \urcorner \circ \Gamma_{A,X}$.

(i) Internal existential quantification preserves $\downarrow$-segments. (cf. chapter 2 s))

(ii) The internal Beck condition applied to the pull back diagram defined by 4).

(iii) The internal Beck condition applied to 3).

Thus $\ulcorner f \urcorner \circ \Gamma_{A,X} \circ F = \ulcorner f_{\restriction} \circ h \urcorner \circ \Gamma_{A,X}$ showing that the fixpoints for F of the form $\ulcorner f \urcorner \circ \Gamma_{A,X}$ are exactly the solutions of our problem.

Sublemma A.1.

If $B \in \mathrm{Hom}_{\underline{\mathbf{E}}}(I, P(A \times X))$ *and* $B \circ F = B$ *then*

$B \circ u_{A,X} \circ F \leqslant B \circ u_{A,X}$.

Sublemma A.2.

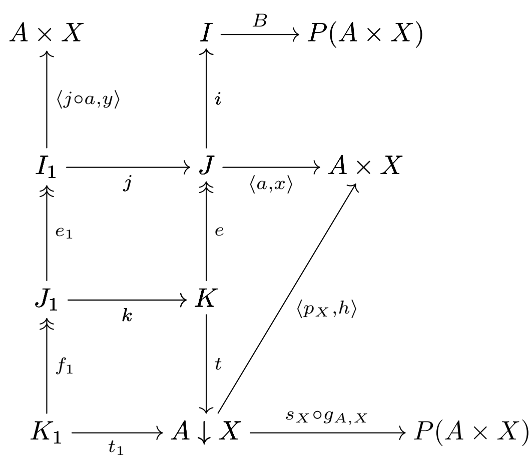
If $B \in \mathrm{Hom}_{\underline{\mathbf{E}}}(I, P(A \times X))$ *then*

$B \circ u_{A,X} \circ F \circ \exists_{p_0} = B \circ u_{A,X} \circ \exists_{p_0} \circ \downarrow seg_{P(A)} \circ P(s)$.

From Sublemma A.1 we get that the smallest fixpoint B for F is in fact a partial graph, i.e. $B = B \circ u_{A,X} = B \circ F$, and substituting this in Sublemma A.2 yields that $B \circ \exists_{p_0}$ is a fixpoint for $\downarrow seg_{P(A)} \circ P(s)$, i.e. $B \circ \exists_{p_0} = \ulcorner true_A \urcorner$ as s is inductive. Thus B is both a partial graph and globally defined. It follows from P2 that $B = \ulcorner f \urcorner \circ \Gamma_{A,X}$ for some (uniquely determined) f.

This concludes the proof of Theorem A.1. □

Proof of Sublemma A.1.



Let $I \in |\underline{\mathbf{E}}|$, $B \in P(A \times X)$ such that $B \circ F = B$. We claim that $B \circ u_{A,X} \circ F \leqslant B \circ u_{A,X}$. Observe that $B \circ u_{A,X} \circ F \leqslant B \circ F = B$ as $u_{A,X} \leqslant id_{P(A \times X)}$.

Let $J \in |\underline{\mathbf{E}}|$, $i \in I$, $a \in A$, $x \in X$ such that $\langle a, x \rangle \underline{\in} i \circ B \circ u_{A,X} \circ F$, and let $I_1 \in |\underline{\mathbf{E}}|$, $j \in J$, $y \in X$ such that $\langle j \circ a, y \rangle \underline{\in} j \circ i \circ B = j \circ i \circ B \circ F$.

We claim that $y = i \circ x$.

From $\langle a, x \rangle \underline{\in} i \circ B \circ u_{A,X} \circ F$ we get $\exists K \in |\underline{\mathbf{E}}|$, $\exists e \in J$ (epi), $\exists t \in A \downarrow X$ such that $t \circ p_X = e \circ a$ and $t \circ h = e \circ x$ and $t \circ s_X \circ g_{A,X} \leqslant e \circ i \circ B \circ u_{A,X}$.

Let $e_1 \circ j = k \circ e$ be a pull back. Then e_1 is epic, and $e_1 \circ \langle j \circ a, y \rangle \underline{\in} e_1 \circ j \circ i \circ B \circ F$ from which we get that $\exists K_1 \in |\underline{\mathbf{E}}|$, $\exists f_1 \in J_1$ (epi), $\exists t_1 \in A \downarrow X$ such that $t_1 \circ p_X = f_1 \circ e_1 \circ j \circ a$, $t_1 \circ h = f_1 \circ e_1 \circ y$ and $t_1 \circ s_X \circ g_{A,X} \leqslant f_1 \circ e_1 \circ j \circ i \circ B$.

Now

i) $(f_1 \circ k \circ t) \circ s_X \circ g_{A,X} \leqslant f_1 \circ k \circ e \circ i \circ B \circ u_{A,X} = f_1 \circ e_1 \circ j \circ i \circ B \circ u_{A,X}$ and

ii) $t_1 \circ s_X \circ g_{A,X} \leqslant f_1 \circ e_1 \circ j \circ i \circ B$ and as

$$(f_1 \circ k \circ t) \circ p_X = f_1 \circ k \circ e \circ a = f_1 \circ e_1 \circ j \circ a = t_1 \circ p_X \quad \left(\underset{def.}{=} \underline{a} \right)$$

it follows that

iii) $(f_1 \circ k \circ t \circ s_X \circ g_{A,X}) \circ \exists_{p_0} = \underline{a} \circ s = (t_1 \circ s_X \circ g_{A,X}) \circ \exists_{p_0}$

It follows from P4 that $f_1 \circ k \circ t \circ s_X \circ g_{A,X} = t_1 \circ s_X \circ g_{A,X}$, and therefore $f_1 \circ k \circ t \circ s_X = t_1 \circ s_X$ as $g_{A,X}$ is monic. But $t_1 \circ p_X = \underline{a} = (f_1 \circ k \circ t) \circ p_X$ and $t_1 \circ s_X = (f_1 \circ k \circ t) \circ s_X$ whence $f_1 \circ k \circ t = t_1$, and so $f_1 \circ e_1 \circ y = t_1 \circ h = f_1 \circ k \circ t \circ h = f_1 \circ k \circ e \circ x = f_1 \circ e_1 \circ j \circ x$ from which we get that $y = j \circ x$ as $f_1 \circ e_1$ is epic.

This proves Sublemma A.1. □

Proof of Sublemma A.2. Let $B \in \mathrm{Hom}_{\underline{\mathbf{E}}}(I, P(A \times X))$, then there exists exactly one $\underline{B} \in \mathrm{Hom}_{\underline{\mathbf{E}}}(I, P(X^A))$ such that $B \circ u_{A,X} \circ \downarrow seg_{P(A \times X)} = \underline{B} \circ \exists_{g_{A,X}}$. This follows from P3 and the fact that $g_{A,X}$ is the equalizer of $u_{A,X}$ and $id_{P(A \times X)}$, and as $u_{A,X}$ is idempotent. Now

$$B \circ u_{A,X} \circ F \circ \exists_{p_0} = B \circ u_{A,X} \circ \downarrow seg_{P(A \times X)} \circ P(g_{A,X}) \circ P(s_X) \circ \exists_{\langle p_X, h \rangle} \circ \exists_{p_0} =$$

$$B \circ u_{A,X} \circ \downarrow seg_{P(A \times X)} \circ P(g_{A,X}) \circ P(s_X) \circ \exists_{p_X} =$$

$$\underline{B} \circ \exists_{g_{A,X}} \circ P(g_{A,X}) \circ P(s_X) \circ \exists_{p_X} = \underline{B} \circ P(s_X) \circ \exists_{p_X} = \underline{B} \circ \exists_{(dom_X)^A} \circ P(s) =$$

$$\underline{B} \circ \exists_{g_{A,X}} \circ \exists_{\exists_{p_0}} \circ P(s) = B \circ u_{A,X} \circ \downarrow seg_{P(A \times X)} \circ \exists_{\exists_{p_0}} \circ P(s) =$$

$$B \circ u_{A,X} \circ \exists_{p_0} \circ \downarrow seg_{P(A)} \circ P(s)$$

This proves Sublemma A.2. □

Theorem A.2. *Inductive relations are preserved by the inverse image functor of a geometric functor on elementary topoi.*

Proof. Let

$$\underline{\mathbf{E}} \underset{R}{\overset{L}{\rightleftarrows}} \underline{\mathbf{E}}_0 \,, \qquad L \dashv R$$

be a geometric functor of elementary topoi, and let $s \in \mathrm{Hom}_{\underline{\mathbf{E}}}(A, P(A))$ be an inductive relation in $\underline{\mathbf{E}}$. We claim that the relation $L(s) \circ \widehat{L}_A \in \mathrm{Hom}_{\underline{\mathbf{E}}_0}(L(A), PL(A))$ is inductive.

If $X \in |\underline{\mathbf{E}}_0|$ and $b \in \mathrm{Hom}_{\underline{\mathbf{E}}_0}(P(X), X)$ we consider the diagram

$$\begin{array}{ccccc}
L(A) & \xrightarrow{L(s)} & LP(A) & \xrightarrow{\widehat{L}_A} & PL(A) \\
\downarrow{\scriptstyle L(f)} & 1 & \downarrow{\scriptstyle L(\exists_f)} & 2 & \downarrow{\scriptstyle \exists_{L(f)}} \\
LR(X) & \xleftarrow{L(c)} & LPR(X) & \xrightarrow{\widehat{L}_{R(X)}} & PLR(X) \\
\downarrow{\scriptstyle v_X} & & 3 & & \downarrow{\scriptstyle \exists_{v_X}} \\
X & & \xleftarrow{\quad b \quad} & & P(X).
\end{array}$$

where $c \in \mathrm{Hom}_{\underline{\mathbf{E}}}(PR(X), R(X))$ is the unique morphism in $\underline{\mathbf{E}}$ making 3 commutative, and $f \in \mathrm{Hom}_{\underline{\mathbf{E}}}(A, R(X))$ such that $s \circ \exists_f \circ c = f$. Finally, 2 is commutative as L preserves epi-mono-factorization and is left exact. This proves that the square is commutative.

Now, any morphism from $L(A)$ to X is of the form $L(g) \circ v_X$ where $g \in \mathrm{Hom}_{\underline{\mathbf{E}}}(A, R(X))$, and if $L(g) \circ v_X = L(s) \circ \widehat{L}_A \circ \exists_{L(g) \circ v_X} \circ b$, then $s \circ \exists_g \circ c = g$ as 2 is commutative for any element in $\mathrm{Hom}_{\underline{\mathbf{E}}}(A, R(X))$ and 3 is commutative by definition of c. As s is inductive we have that $f = g$.

This proves that $L(s) \circ \widehat{L}_A$ is inductive and concludes the proof of Theorem A.2. □

Theorem A.2 allows us to give the following characterization of inductive relations:

If $s \in \mathrm{Hom}_{\underline{\mathbf{E}}}(A, P(A))$ then s is inductive iff the following diagram is a pull back.

$$\begin{array}{ccccc} P(A) & \xrightarrow{\langle \downarrow seg \circ P(s), id_{P(A)} \rangle} & P(A) \times P(A) & \xrightarrow{\uparrow \widehat{seg}_A} & \Omega \\ \Big\uparrow{\scriptstyle \ulcorner true_A \urcorner} & & \text{(I)} & & \Big\uparrow{\scriptstyle true} \\ 1 & & \xrightarrow{id_1} & & 1 \end{array}$$

Indeed, we have seen that s is inductive iff (I) is a pull back for global sections.

By Theorem A.2 we know that the functors $(\ ) \times Z : \underline{\mathbf{E}} \longrightarrow \underline{\mathbf{E}}/Z$ preserve inductive relations, and as they are logical they preserve the diagram (I). But these facts imply that (I) is a pull back in $\underline{\mathbf{E}}$.

Corollary A.1. *Logical functors on elementary topoi preserve inductive relations.*

<u>Remarks</u>.

i). Inductive relations are irreflexive and acyclic (indeed, the transitive hull of an inductive relation is inductive, and therefore inductive).

ii). In any elementary topos we have the notion of a well ordered object, i.e. an internally ordered object such that any non-empty subobject of the object has a smallest element. In case we are in an elementary Boolean topos the classical proof that a well ordering is in fact a linear ordering whose strict $\downarrow$-segment satisfies the principle of transfinite induction is easily seen to be valid, (There is a constructive proof of this fact!). As an inverse image functor preserves inductive relations and linearity of internal orderings, it follows that in the case of elementary Boolean topoi an inverse image functor preserves well orderings.

This result was proved by W. Mitchell in the case of elementary Boolean topoi in which support splits (BT') by another method, and conjectured to be valid for all elementary Boolean topoi (BT) [20].

Appendix B

Impredicative Constructions in Elementary Topoi

This appendix is intended to be an example of how to derive information from impredicative constructions in elementary topoi. Formally, however, it supplies the proofs of two theorems which were announced in "Some Topos Theoretic Concepts of Finiteness", [8], to which we refer the reader for additional information.

Let us first give an example from the category of sets explaining what we mean by predicative / impredicative.

<u>The predicative approach</u>.

Let G be a group and let A be a subset of G, then we know that the subgroup $\underline{\mathrm{A}}$ of G generated by A is given by

$$\underline{A} = \{\prod_{i=1}^{n} a_i | a_i \in A \cup A^{-1}, n \in \mathbb{N}\}.$$

Assume that A is a commutative subset of G, i.e. $\forall a, b \in A : a \cdot b = b \cdot a$. We see, by induction on $n \in \mathbb{N}$, that $\underline{\mathrm{A}}$ is a commutative subgroup of G.

<u>The impredicative approach</u>.

Consider the same problem once more. $\underline{\mathrm{A}}$ can be described by

$$\underline{A} = \bigcap \{H \subseteq G | H \text{ is a subgroup of } G \text{ and } A \subseteq H\}.$$

Consider the centralizer $C(A) = \{b \in G | \forall a \in A : a \cdot b = b \cdot a\}$ of A. $C(A)$ is a subgroup of G, and A is contained in $C(A)$ iff A is a commutative subset of G. Now $A \subseteq C(A) \cap CC(A)$, but $C(A) \cap CC(A)$ is a commutative subgroup of G and as it contains A it must contain the smallest subgroup of G containing A. I.e. $\underline{A} \subseteq C(A) \cap CC(A)$. It follows that $\underline{\mathrm{A}}$ is a commutative subgroup of G.

We leave to the reader to verify for himself that the first proof can be lifted to any elementary topos with a natural number object, whereas the second proof can be lifted to all elementary topoi.

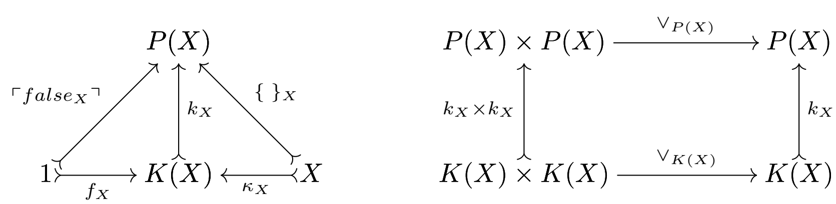
Let $\underline{\mathbf{E}}$ be a fixed elementary topos. By the fixpoint theorem we know how to perform impredicative constructions (such as $\underline{\text{A}}$) in $\underline{\mathbf{E}}$. Now the very construction is fundamental, but in itself it is not enough supply the proof of the problem (such as the commutativity of $\underline{\text{A}}$) for which the construction was performed. Fortunately, however, the fixpoint theorem yields the additional information that we have a smallest fixpoint, and it is the understanding of this fact that allows us to draw conclusions (such as the commutativity of $\underline{\text{A}}$).

We shall now give a rather general example of how these ideas can be applied in $\underline{\mathbf{E}}$.

From Manes' theorem for elementary topoi we know that the algebras for the internal power monad $\exists$ on $\underline{\mathbf{E}}$ are the complete lattices in $\underline{\mathbf{E}}$. So, what is the "finitary" part of $\exists$? Explicitly, given X in $\underline{\mathbf{E}}$ what is the subobject of $P(X)$ generated by $\{\ \}_X$, $\ulcorner false_X \urcorner$ and $\vee_{P(X)}$?

In the category of sets we know from W. Sierpiński that the subset asked for is the set of all "finite" subsets of X, where "finite" refers to the lattice theoretic concept of finiteness which serves mathematics in the topological formulation of compactness.

Given X in $\underline{\mathbf{E}}$ let $k_X : K(X) \rightarrowtail P(X)$ be the smallest subobject of $P(X)$ which contains $\{\ \}_X$, $\ulcorner false_X \urcorner$ and $\vee_{P(X)}$, i. e. such that

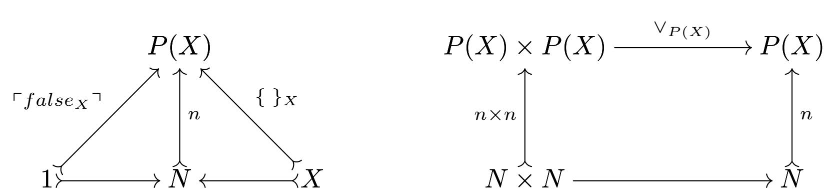


the indicated factorizations exist.

The existence of $K(X)$ is guaranteed by the fixpoint theorem. Explicitly by applying the following internal functors on $PP(X)$:

$$(*) \qquad \begin{cases} \text{“}\textit{adding the singletons}\text{”} = \langle id_{PP(X)}, !_{PP(X)} \circ \ulcorner s_X \urcorner \rangle \circ \vee_{PP(X)} \\ \text{“}\textit{adding false}\text{”} = \langle id_{PP(X)}, !_{PP(X)} \circ \ulcorner false_X \urcorner \circ \{\ \}_{P(X)} \rangle \circ \vee_{PP(X)} \\ \text{“}\textit{closing up under binary union}\text{”} = \Delta_{PP(X)} \circ \tilde{p}_{P(X),P(X)} \circ \exists_{\vee_{P(X)}}. \end{cases}$$

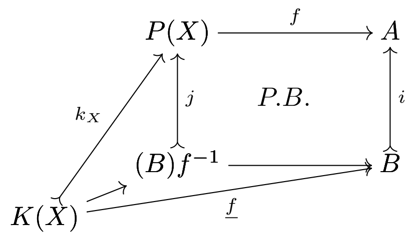
A fixpoint for $(*)$ is a subobject $n : N \rightarrowtail P(X)$ such that



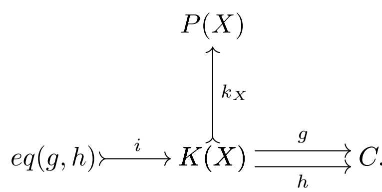
the indicated factorization exist. The fixpoint theorem says that there is a smallest such subobject, namely k_X.

In a situation like this we have the following two principles:

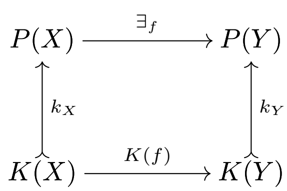
The mapping principle. If $f \in \mathrm{Hom}_{\underline{\mathbf{E}}}(P(X), A)$ and $i : B \rightarrowtail A$ (monic) are morphisms in $\underline{\mathbf{E}}$ and if $j : (B)f^{-1} \rightarrowtail P(X)$ is a fixpoint for $(*)$, then there exists a morphism $\underline{f} \in \mathrm{Hom}_{\underline{E}}(K(X), B)$ such that $\underline{f} \circ i = k_X \circ f$.



The uniqueness principle. If $g, h \in \mathrm{Hom}_{\underline{E}}(K(X), C)$ and if $i = eq(g,h) \rightarrowtail K(X)$ then $g = h$ provided $i \circ k_X$ is a fixpoint for $(*)$.



If $f \in \mathrm{Hom}_{\underline{E}}(X, Y)$ then as $\ulcorner false_X \urcorner \circ \exists_f = \ulcorner false_Y \urcorner$, $\{\ \}_X \circ \exists_f = f \circ \{\ \}_Y$ and $\vee_{P(X)} \circ \exists_f = \exists_f \times \exists_f \circ \vee_{P(Y)}$ we get from the mapping principle (applied to $\exists_f$ and k_X) that there exists a factorization:



It follows that the assignment $X \mapsto K(X)$ and $f \mapsto K(f)$ defines a functor

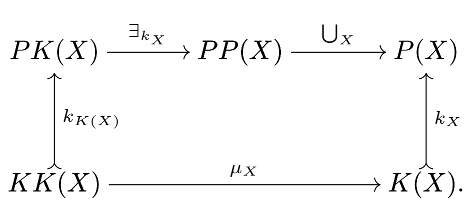
$K : \underline{\mathbf{E}} \longrightarrow \underline{\mathbf{E}}$ that

$\kappa = \{\kappa_X : X \rightarrowtail K(X)\}_{X \in |\underline{\mathbf{E}}|}$ and

$k = \{k_X : K(X) \rightarrowtail P(X)\}_{X \in |\underline{\mathbf{E}}|}$

define pointwise monic natural transformations, $\kappa : id_{\underline{\mathbf{E}}} \Rightarrow K$, $k : K \Rightarrow \exists$ and $\kappa \circ k = \{\ \}$.

Consider $\exists_{k_X} \circ \bigcup_X$. As this internal functor has a right adjoint, namely $\downarrow seg_{P(X)} \circ P(k_X)$ and as $\{\ \}_{K(X)} \circ \exists_{k_X} \circ \bigcup_X = k_X$ we get from the mapping principle that there exists a factorization:



As k is pointwise monic and natural it follows that

$$\mu = \{\mu_X : KK(X) \longrightarrow K(X)\}_{X \in |\underline{\mathbf{E}}|}$$

defines a natural transformation $\mu : KK \Rightarrow K$ such that $\mathbb{K} = (K, \kappa.\mu)$ is a monad on $\underline{\mathbf{E}}$ and such that $k : \mathbb{K} \Rightarrow \mathbb{\exists}$ is a transformation of monads.

Let $C_0(\underline{\mathbf{E}})$ be the category of upper semilattices in $\underline{\mathbf{E}}$ with smallest global section, and right exact morphisms.

Proposition B.1. *$C_0(\underline{\mathbf{E}})$ is isomorphic to $\underline{\mathbf{E}}^{\mathbb{K}}$.*

Proof. The functor $I : \underline{\mathbf{E}}^{\mathbb{K}} \longrightarrow C_0(\underline{\mathbf{E}})$ is defined as follows: $I(A, \xi : K(A) \to A) = (A, \vee, 0)$, $I(f) = f$ where $\vee = \kappa_A \times \kappa_A \circ \vee_{P(A)} \circ \xi$ and $0 = f_A \circ \xi$.

The functor $J : C_0(\underline{\mathbf{E}}) \longrightarrow \underline{\mathbf{E}}^{\mathbb{K}}$ is constructed in the following way.

If $(A, \vee, 0) \in |C_0(\underline{\mathbf{E}}|$ we construct the extension of those subobjects of A which have a sup in A as the inverse image of $\uparrow\! seg_A$ along $\uparrow\! seg_{P(A)} \circ P(\downarrow\! seg_A)$. As this extension is readily seen to be a fixpoint for $(*)$, by the mapping principle we get the following factorization:

$$\begin{array}{ccccc}
P(A) & \xrightarrow{\uparrow seg_{P(A)}} & PP(A) & \xrightarrow{P(\downarrow seg_A)} & P(X) \\
\uparrow {\scriptstyle k_A} & & & & \uparrow {\scriptstyle k_X} \\
K(A) & & \xrightarrow{\quad\quad \xi_\vee \quad\quad} & & A
\end{array}$$

This leads to the definition $J(A, \vee, 0) = (A, \xi_\vee)$, $J(f) = f$.

The fact that I is a well defined functor is a purely diagrammatic proof, recalling that $\vee$ is to be idempotent, associative and commutative. 0 is zero as $\ulcorner false_A \urcorner$ is zero. Morphisms are obvious.

The fact that J is a well defined functor follows as $\xi_\vee$ is constructed out of "the sup of subobjects having a sup", and the uniqueness principle applied to the appropriate morphisms. Likewise for morphisms.

The passage $(A, \vee, 0) \mapsto (A, \xi_\vee) \mapsto (A, \vee, 0)$ is the identity as $\xi_\vee$ is the restriction of sups.

The passage $(A, \xi) \mapsto (A, \vee, 0) \mapsto (A, \xi_\vee)$ is the identity by the uniqueness principle. $\square$

Recall that the cotensorial strength

$$\lambda = \{\lambda_{X,Y} : P(Y^X) \longrightarrow P(Y)^X\}_{(X,Y) \in |\underline{\mathbf{E}}| \times |\underline{\mathbf{E}}|}$$

for ∃I is a pointwise co-continuous internal functor preserving singletons in the following sense $\{\ \}_{X^Y} \circ \lambda_{X,Y} = (\{\ \}_Y)^X$. Thus from the mapping principle (applied to $\lambda_{X,Y}$ and $(k_Y)^X$) we get a new factorization:

$$\begin{array}{ccc} P(Y^X) & \xrightarrow{\lambda_{X,Y}} & P(Y)^X \\ \uparrow{\scriptstyle k_{Y^X}} & & \uparrow{\scriptstyle (k_Y)^X} \\ K(Y^X) & \xrightarrow{\overline{\lambda}_{X,Y}} & K(Y)^X \end{array}$$

$\overline{\lambda}$ extends to a cotensorial strength for K, and as the associated tensorial strengths for K commute with k, and ∃I is symmetric, we get that IK is a symmetric monoidal sub- (via k) monad of ∃I In particular we have a monoidal transformation $\ddot{p}$ making the following diagram commutative:

$$\begin{array}{ccc} P(X) \times P(Y) & \xrightarrow{\tilde{p}_{X,Y}} & P(X \times Y) \\ \uparrow{\scriptstyle k_X \times k_Y} & & \uparrow{\scriptstyle k_{X \times Y}} \\ K(X) \times K(Y) & \xrightarrow{\ddot{p}_{X,Y}} & K(X \times Y) \end{array}$$

Definition B.1 (Sierpiński [23])**.** *An object X in $\underline{\mathbf{E}}$ is said to be K-finite iff $\ulcorner true_X \urcorner$ belongs to $K(X)$, i.e. iff there is a factorization:*

$$\begin{array}{ccc} & & P(X) \\ & \nearrow{\scriptstyle \ulcorner true_X \urcorner} & \uparrow{\scriptstyle k_X} \\ 1 & \xrightarrow[t_X]{} & K(X) \end{array}$$

(Actually in Sierpiński's original definition on page 106 in [23] finite sets are non-empty.)

The following remarks are contained in [8], though the proofs given here are of another nature.

1). The initial object ∅ is K-finite.

Proof. $P(\varnothing)$ is the terminal object, whence $\ulcorner true_\varnothing \urcorner = \ulcorner false_\varnothing \urcorner$. □

2). The terminal object 1 is K-finite, and $K(1) = 2$.

Proof. $\{\ \}_1 = \ulcorner true \urcorner$. $2 \rightarrowtail \Omega$ is closed under binary union and generated by $true$ and $false$. □

3). If X is K-finite and $f : X \twoheadrightarrow Y$ is epic, then Y is K-finite.

Proof. As $\ulcorner true_X \urcorner \circ \exists_f = \ulcorner true_Y \urcorner$ as f is epic, we have that $t_X \circ K(f)$ is the proof that Y is K-finite. □

4). If X and Y are K-finite, then $X \times Y$ is K-finite.

Proof. As $\langle \ulcorner true_X \urcorner, \ulcorner true_Y \urcorner \rangle \circ \tilde{p}_{X,Y} = \ulcorner true_{X \times Y} \urcorner$ we have that $\langle t_X, t_Y \rangle \circ \ddot{p}_{X,Y}$ is the proof that $X \times Y$ is K-finite. □

5). If $K(X)$ is K-finite then X is K-finite.

Proof. As $\exists_{\kappa_X} \circ \exists_{k_X} \circ \bigcup_X = id_{P(X)}$ we have that $\ulcorner true_X \urcorner = \ulcorner true_{K(X)} \urcorner \circ \exists_{k_X} \circ \bigcup_X$, and therefore $t_{K(X)} \circ \mu_X$ is the proof that X is K-finite. □

6). We leave to the reader the pleasure of discovering a proof of the fact that the coproduct of two objects in $\underline{\mathbf{E}}$ is K-finite iff each of the objects is K-finite. (There is an alternative proof in [8]).

7). As a consequence of 2) and 6) we see that $K(1) = 2 = 1 + 1$ is K-finite.

As for the proofs of 1) - 7), they are only given here in order to illustrate how to derive information from the impredicative construction K. In [8], it is shown that K-finiteness has a predicative description, and as this description is valid in any elementary topos, it is clear that we should profit from this description. (Cf. Theorem B.2 below).

Theorem B.1. *The inverse image functor of a geometric functor on elementary topoi preserves K-finite objects.*

Proof. Let

$$\underline{\mathbf{E}} \underset{R}{\overset{L}{\rightleftarrows}} \underline{\mathbf{E}}_0 \,, \qquad L \dashv R$$

be a geometric functor of elementary topoi.

If $X \in |\underline{\mathbf{E}}|$ we claim that there is a morphism l_X from $LK(X)$ to $KL(X)$ such that

$$\begin{array}{ccc} LP(X) & \xrightarrow{\widehat{L}_X} & PL(X) \\ \uparrow{\scriptstyle L(k_X)} & & \uparrow{\scriptstyle k_{L(X)}} \\ LK(X) & \xrightarrow{\bar{l}_X} & KL(X) \end{array}$$

is commutative.

Recall that the adjoint of $\widehat{L}_X$, $t_{P(X)} \circ R(\widehat{L}_X) : P(X) \longrightarrow RPL(X)$, is an internal functor having a right adjoint. As $\{\ \}_X \circ t_{P(X)} \circ R(\widehat{L}_X) = t_X \circ R(\{\ \}_X)$, it follows from the mapping principle (applied to $t_{P(X)} \circ R(\widehat{L}_X)$ and $R(k_{L(X)})$) that there exists a factorization:

$$\begin{array}{ccc} P(X) & \xrightarrow{t_{P(X)} \circ R(\widehat{L}_X)} & RPL(X) \\ \uparrow{\scriptstyle k_X} & & \uparrow{\scriptstyle R(k_{L(X)})} \\ K(X) & \longrightarrow & RKL(X) \end{array}$$

This proves the assertion on the existence of l_X.

As $L(\ulcorner true_X \urcorner) \circ \widehat{L}_X = \ulcorner true_{L(X)} \urcorner$, it follows that if X is K-finite then $L(t_X) \circ l_X$ is the proof that $L(X)$ is K-finite.

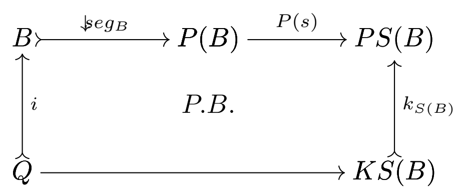
This concludes the proof of Theorem B.1. □

Remark. In the notation of Theorem B.1 we have that l_X is an epimorphism. Indeed, as $L(\{\ \}_X) \circ \widehat{L}_X = \{\ \}_{L(X)}$, and as l_X is right exact, it follow that the image of l_X considered as a subobject of $PL(X)$ is a fixpoint for $(*)$. This shows that l_X is epic. If L preserves universal quantification then l_X is iso.

The final section contains a number of terms undefined in this work. They are all taken from [8] to which we refer the reader for the definitions and theorems applied.

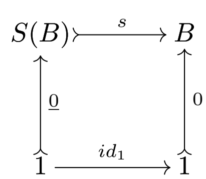
Let B be a complete Heyting algebra in $\underline{\mathbf{E}}$, and assume that B is an algebraic lattice in $\underline{\mathbf{E}}$, and let $s : S(B) \rightarrowtail B$ be the extension of the intranscessible elements in B.

Consider the pull back diagram:

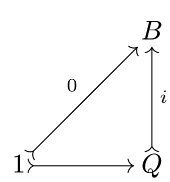


We claim that the subobject $i : Q \rightarrowtail B$ is a $C_0(\underline{\mathbf{E}})$-subobject of B, and that the extension $i_B : T(B) \rightarrowtail B$ of the atoms of B is contained in Q.

1). Let $0 : 1 \rightarrowtail B$ be the smallest global section in B. As 0 is intranscessible we have the following commutative diagram (pull back)



Now $0 \circ \downarrow seg_B \circ P(s) = 0 \circ \{\ \}_B \circ P(s) = \{\ \}_1 \circ \exists_0 \circ P(s) = \{\ \}_1 \circ \exists_{\underline{0}} = \underline{0} \circ \{\ \}_{S(B)}$. (Using $0 \dashv !_B$, and the internal Beck condition applied to the square). It follows that 0 factors through Q, i.e. we have a commutative diagram:



2). Consider the following diagram:

$$\begin{array}{ccccc}
B \times B & \xrightarrow{\downarrow seg_B \times \downarrow seg_B} & P(B) \times P(B) & \xrightarrow{P(s)\times P(s)} & PS(B) \times PS(B) \\
 & & \downarrow{\scriptstyle \tilde{p}_{B,B}} & \text{II} & \downarrow{\scriptstyle \tilde{p}_{S(B),S(B)}} \\
\downarrow{\scriptstyle \vee} & \text{I} & P(B \times B) & \xrightarrow{P(s\times s)} & P(S(B) \times S(B)) \\
 & & \downarrow{\scriptstyle \exists_\vee} & \not\nearrow & \downarrow{\scriptstyle \exists_\vee} \\
B & \xrightarrow[\downarrow seg_B]{} & P(B) & \xrightarrow[P(s)]{} & PS(B)
\end{array}$$

The square I is commutative as B is a distributive lattice. The square II is commutative by the naturality of $\tilde{p}$ with respect to P. The inequality in the last square follows from the fact that $S(B)$ is an upper-sub-semilattice of B.

We claim that the outer square is commutative.

Let a_0, a_1 be elements in B and let c be an intranscessible element in B such that $c \leqslant a_0 \vee a_1$. The trick is to notice that $c_i = c \wedge a_i$ is the sup of a family A_i of intranscessible elements below a_i. It follows, by distributivity, that c is the sup of $A_0 \underline{\vee} A_1$ as c is intranscessible.

This shows the asserted commutativity. It follows that we have a commutative diagram

$$\begin{array}{ccc}
B \times B & \xrightarrow{\vee} & B \\
\uparrow{\scriptstyle i\times i} & & \uparrow{\scriptstyle i} \\
Q \times Q & \longrightarrow & Q
\end{array}$$

3). Let $a : 1 \longrightarrow B$ be an atom in B (defined over 1).
As $(a) \downarrow seg_B \simeq \Omega \simeq P(1)$, we have that $S((a) \downarrow seg_B) \simeq SP(1) = K(1) = 2$, which is K-finite. Thus a factors through Q.

Applying this argument to the atom $\langle i_B, id_{T(B)} \rangle : T(B) \rightarrowtail B \times T(B)$ in the topos $\underline{\mathbf{E}}/T(B)$ yields that $i_B : T(B) \rightarrowtail B$ is contained in Q.

Recalling that $SP(X) = K(X)$ and that the atoms in $P(X)$ are the singletons we get Q is a fixpoint for $(*)$ in the case $B = P(X)$. Thus there exists a morphism $\sigma eg_X : K(X) \rightarrowtail KK(X)$ such that

$$\begin{array}{ccccc}
P(X) & \xrightarrow{\downarrow seg_{P(X)}} & PP(X) & \xrightarrow{P(k_X)} & PK(X) \\
\uparrow{\scriptstyle k_X} & & & & \uparrow{\scriptstyle k_{K(X)}} \\
K(X) & & \xrightarrow{\sigma eg_X} & & KK(X)
\end{array}$$

is commutative.

If X is K-finite. then $t_X \circ \sigma eg_X$ is the proof that $K(X)$ is K-finite as $\ulcorner true_X \urcorner \circ \downarrow seg_{P(X)} \circ P(k_X) = \ulcorner true_{K(X)} \urcorner$.

Taking into account these remarks and 5) above we record

Theorem B.2. *An object X in an elementary topos $\underline{\mathbf{E}}$ is K-finite iff $K(X)$ is K-finite.*

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